p-adic asymptotic properties of integer sequences

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Fibonacci sequence

$F(n)_{n\geq 0}$: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$

$$F(n) = F(n-1) + F(n-2)$$

$(F(n) \bmod m)_{n\geq 0}$ is periodic.

$(F(n) \bmod 2)_{n\geq 0}$: $0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$

$(F(n) \bmod 4)_{n\geq 0}$: $0, 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, 0, 1, 1, 2, \ldots$

$(F(n) \bmod 8)_{n\geq 0}$: $0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, 1, 2, \ldots$

only attains $\frac{6}{8}$ of all residues

$(F(n) \bmod 16)_{n\geq 0}$: $0, 1, 1, 2, 3, 5, 8, 13, 5, 2, 7, 9, 0, 9, 9, 2, \ldots$

attains $\frac{11}{16}$ of all residues

Density modulo 2^α: $1, 1, 1, \frac{6}{8}, \frac{11}{16}, \frac{21}{32}, \frac{21}{32}, \frac{21}{32}, \frac{21}{32}, \frac{21}{32}, \ldots$ stable at $\frac{21}{32}$?

Limiting density

What is the limiting density $\lim_{\alpha \to \infty} \frac{|\{F(n) \bmod p^\alpha : n \geq 0\}|}{p^\alpha}$?

This limit exists.

Density modulo 2^α: $1, 1, 1, \frac{6}{8}, \frac{11}{16}, \frac{21}{32}, \frac{21}{32}, \frac{21}{32}, \frac{21}{32}, \frac{21}{32}, \dots$

Burr (1971): $F(n)$ attains all residues modulo 3^α and 5^α.

Density modulo 11^α: $1, \frac{7}{11}, \frac{67}{121}, \frac{732}{1331}, \frac{8042}{14641}, \frac{88457}{161051}, \dots$
$\approx 1., .63636, .55372, .54996, .54928, .54925, \dots$

Theorem (Rowland–Yassawi 2017)

For $p = 11$ the limiting density is $\frac{145}{264}$.

Structure in $F(2^n)$

$$\begin{aligned}
F(1) &= 1 = 1_2\\
F(2) &= 1 = 1_2\\
F(4) &= 3 = 11_2\\
F(8) &= 21 = 10101_2\\
F(16) &= 987 = 1111011011_2\\
F(32) &= 2178309 = 1000010011110100000101_2
\end{aligned}$$

Theorem (Rowland–Yassawi 2017)

The limits $\lim_{n\to\infty} F(2^{2n})$ *and* $\lim_{n\to\infty} F(2^{2n+1})$ *are equal to* $\pm\sqrt{-\frac{3}{5}}$ *in* $\mathbb{Z}_2$.

Two limits

Values of $F(2^{2n})$:

Values of $F(2^{2n+1})$:

Subtract the limits

Values of $F(2^{2n}) - \lim_{m\to\infty} F(2^{2m})$:

Values of $F(2^{2n+1}) - \lim_{m\to\infty} F(2^{2m+1})$:

Shift

Values of $\frac{F(2^{2n}) - \lim_{m\to\infty} F(2^{2m})}{2^{2n}}$:

Values of $\frac{F(2^{2n+1}) - \lim_{m\to\infty} F(2^{2m+1})}{2^{2n+1}}$:

These pictures suggest two 2-adic power series:

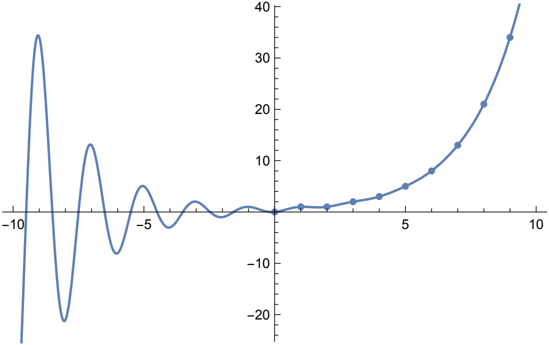
$$F(x) = \quad a_0 + a_1 x + \cdots \qquad \text{for } x = 2^{2n}$$

$$F(x) = -a_0 + a_1 x + \cdots \qquad \text{for } x = 2^{2n+1}$$

Interpolation to $\mathbb{R}$

Let $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$. Binet's formula:

$$F(n) = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}$$



$$F(x) = \frac{\exp(x \log \phi) - \cos(\pi x) \exp(x \log(-\bar{\phi}))}{\sqrt{5}}$$

Twisted interpolation to $\mathbb{Z}_3$

Let $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$ in $\mathbb{Q}_3(\sqrt{5})$.
Let $\omega(\phi), \omega(\bar{\phi}) \in \mathbb{Q}_3(\sqrt{5})$ be 8th roots of unity congruent to $\phi, \bar{\phi}$ mod 3.

Theorem (Rowland–Yassawi 2017)

For each $0 \le i \le 7$*, define the function* $F_i : \mathbb{Z}_3 \to \mathbb{Z}_3$ *by*

$$F_i(x) := \frac{\omega(\phi)^i \exp_3\left(x \log_3 \frac{\phi}{\omega(\phi)}\right) - \omega(\bar{\phi})^i \exp_3\left(x \log_3 \frac{\bar{\phi}}{\omega(\bar{\phi})}\right)}{\sqrt{5}}.$$

is the unique continuous function s.t. $F(n) = F_i(n)$ *for all* $n \equiv i \mod 8$*.*

Since $3^{2n} \equiv 1 \mod 8$,

$$\lim_{n\to\infty} F(3^{2n}) = \lim_{n\to\infty} F_1(3^{2n}) = F_1(0) = \frac{\omega(\phi) - \omega(\bar{\phi})}{\sqrt{5}} = \pm\sqrt{\frac{2}{5}}.$$

Limits

$p = 2$:

$p = 3$:

Because convergence is "backward" in $\mathbb{Z}_p$, truncating a power series gives correct least-significant digits.
This is good for number theory (congruence modulo p^α).

$p = 11$:

Extensions of $\mathbb{Q}$

- If $p \equiv 1$ or $4 \mod 5$, then $x^2 = 5$ has solutions in $\mathbb{Z}_p$.
- If $p \equiv 2$ or $3 \mod 5$, then $x^2 = 5$ has no solutions in $\mathbb{Z}_p$.
  $\implies$ Work in $\mathbb{Q}_p(\sqrt{5})$.
- If $p = 5$, work in $\mathbb{Q}_5(\sqrt{5})$.

Density of residues modulo 11^α

Let μ be the Haar measure on $\mathbb{Z}_p$ defined by $\mu(m + p^\alpha \mathbb{Z}_p) = \frac{1}{p^\alpha}$.

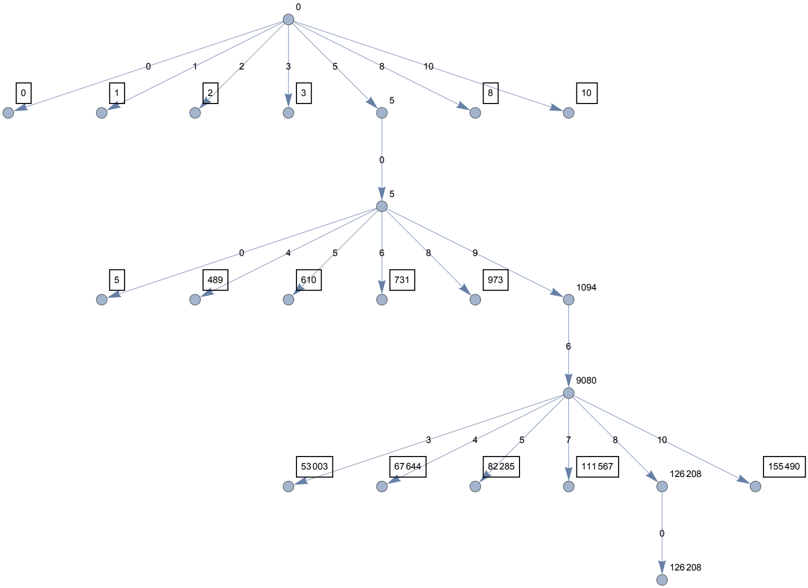
Theorem (Rowland–Yassawi 2017)

The limiting density of residues attained by the Fibonacci sequence modulo 11^α *is*

$$\lim_{\alpha \to \infty} \frac{|\{F(n) \bmod 11^\alpha : n \geq 0\}|}{11^\alpha} = \mu\left(\bigcup_{i=0}^{9} F_i(\mathbb{Z}_{11})\right) = \frac{145}{264}.$$

The twisted interpolation of $F(n)$ to $\mathbb{Z}_{11}$ consists of 10 functions $F_0(x), \ldots, F_9(x)$.

Residues modulo 11^α



Constant-recursive sequences

Let $s(n)_{n \geq 0}$ be a sequence of p-adic integers satisfying a recurrence

$$s(n+\ell) + a_{\ell-1} s(n+\ell-1) + \cdots + a_1 s(n+1) + a_0 s(n) = 0$$

with constant coefficients $a_i \in \mathbb{Z}_p$.

Theorem (Rowland–Yassawi 2017)

$s(n)_{n \geq 0}$ has an approximate twisted interpolation to $\mathbb{Z}_p$. That is, there exists q a power of p, a finite partition $\mathbb{N} = \bigcup_{j \in J} A_j$ with each A_j dense in $r + q\mathbb{Z}_p$ for some $0 \leq r \leq q-1$, finitely many continuous functions $s_j : \mathbb{Z}_p \to K$, and non-negative constants C, D with $D < 1$ such that

$$|s(n) - s_j(n)|_p \leq C \cdot D^n$$

for all $n \in A_j$ and $j \in J$.