Wholesale congruences for sequences arising in combinatorics

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Joint work with Reem Yassawi

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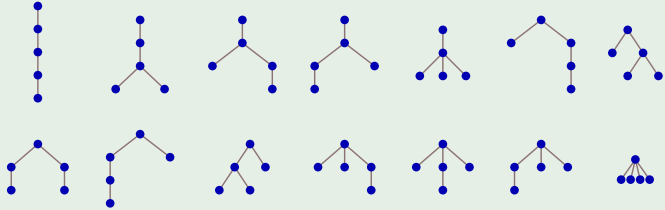
What do combinatorial sequences look like modulo a prime power?

Example

How many plane trees have *n* edges?



$C(4) = 14$

Catalan numbers:
$C(n)_{n \geq 0} = 1, 1, 2, 5, 14, 42, 132, 429, \dots$

Modulo 2: $1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, \dots$

$C(n)$ is odd if and only if $n + 1$ is a power of 2.
(follows from Kummer 1852)

Catalan numbers modulo 4: $1, 1, 2, 1, 2, 2, 0, 1, 2, 2, 0, 2, 0, 0, 0, 1, \dots$

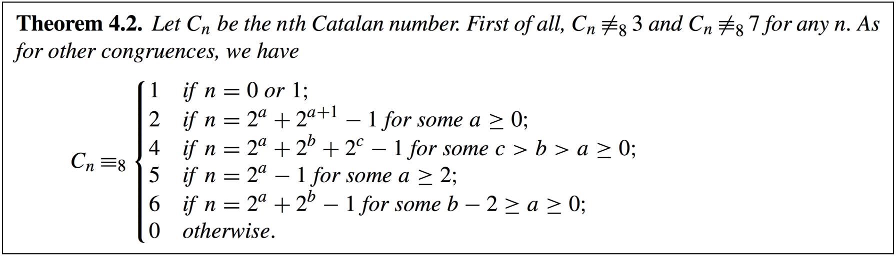
Theorem (Eu–Liu–Yeh 2008)

For all $n \geq 0$,

$$C(n) \bmod 4 = \begin{cases} 1 & \text{if } n+1 = 2^a \text{ for some } a \geq 0 \\ 2 & \text{if } n+1 = 2^b + 2^a \text{ for some } b > a \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $C(n) \not\equiv 3 \mod 4$.

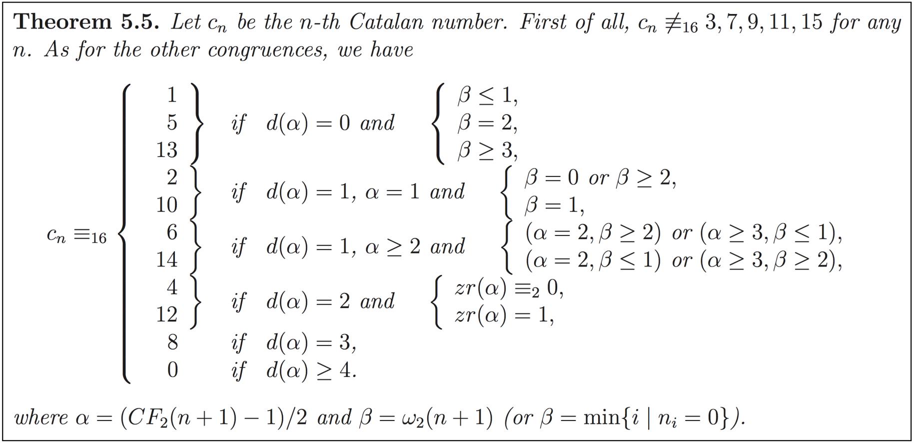
Catalan numbers modulo 8: $1, 1, 2, 5, 6, 2, 4, 5, 6, 6, 4, 2, 4, 4, 0, 5, \dots$



Theorem 4.2. *Let C_n be the nth Catalan number. First of all, $C_n \not\equiv_8 3$ and $C_n \not\equiv_8 7$ for any n. As for other congruences, we have*

$$C_n \equiv_8 \begin{cases} 1 & \text{if } n = 0 \text{ or } 1; \\ 2 & \text{if } n = 2^a + 2^{a+1} - 1 \text{ for some } a \geq 0; \\ 4 & \text{if } n = 2^a + 2^b + 2^c - 1 \text{ for some } c > b > a \geq 0; \\ 5 & \text{if } n = 2^a - 1 \text{ for some } a \geq 2; \\ 6 & \text{if } n = 2^a + 2^b - 1 \text{ for some } b - 2 \geq a \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Liu and Yeh (2010) determined $C(n) \bmod 16$:



Theorem 5.5. *Let c_n be the n-th Catalan number. First of all, $c_n \not\equiv_{16} 3, 7, 9, 11, 15$ for any n. As for the other congruences, we have*

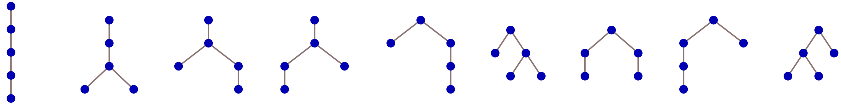
$$c_n \equiv_{16} \begin{cases} \left.\begin{array}{c}1\\ 5\\ 13\end{array}\right\} & \textit{if} \quad d(\alpha) = 0 \ \textit{and} \quad \begin{cases} \beta \le 1, \\ \beta = 2, \\ \beta \ge 3, \end{cases} \\ \left.\begin{array}{c}2\\ 10\end{array}\right\} & \textit{if} \quad d(\alpha) = 1,\ \alpha = 1 \ \textit{and} \quad \begin{cases} \beta = 0 \ \textit{or} \ \beta \ge 2, \\ \beta = 1, \end{cases} \\ \left.\begin{array}{c}6\\ 14\end{array}\right\} & \textit{if} \quad d(\alpha) = 1,\ \alpha \ge 2 \ \textit{and} \quad \begin{cases} (\alpha = 2, \beta \ge 2) \ \textit{or} \ (\alpha \ge 3, \beta \le 1), \\ (\alpha = 2, \beta \le 1) \ \textit{or} \ (\alpha \ge 3, \beta \ge 2), \end{cases} \\ \left.\begin{array}{c}4\\ 12\end{array}\right\} & \textit{if} \quad d(\alpha) = 2 \ \textit{and} \quad \begin{cases} zr(\alpha) \equiv_2 0, \\ zr(\alpha) = 1, \end{cases} \\ 8 & \textit{if} \quad d(\alpha) = 3, \\ 0 & \textit{if} \quad d(\alpha) \ge 4. \end{cases}$$

where $\alpha = (CF_2(n+1) - 1)/2$ and $\beta = \omega_2(n+1)$ (or $\beta = \min\{i \mid n_i = 0\}$).

They also determined $C(n) \bmod 64$.

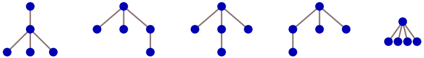
What is the right framework?

How many plane trees with n edges have the property that each vertex has at most 2 children?



$M(4) = 9$

Excluded:



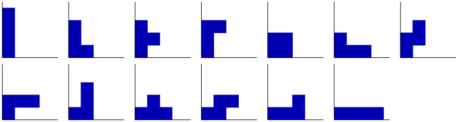
Motzkin numbers:
$M(n)_{n \geq 0} = 1, 1, 2, 4, 9, 21, 51, 127, \ldots$

Modulo 8: $1, 1, 2, 4, 1, 5, 3, 7, 3, 3, 4, 6, 7, 3, 2, 4, \ldots$

Theorem (Eu–Liu–Yeh; conj. by Deutsch–Sagan–Amdeberhan)

$M(n) \not\equiv 0 \mod 8$ *for all* $n \geq 0$.

Number of directed animals: $P(n)_{n \geq 0} = 1, 1, 2, 5, 13, 35, 96, 267, \ldots$



$P(4) = 13$

Number of restricted hexagonal polyominoes:
$H(n)_{n \geq 0} = 1, 1, 3, 10, 36, 137, 543, 2219, \ldots$

Riordan numbers: $R(n)_{n \geq 0} = 1, 0, 1, 1, 3, 6, 15, 36, \ldots$

Theorem (Deutsch–Sagan 2006)

There exists a set $C = \{1, 3, 4, 5, 7, \ldots\}$ with the property that

- *$P(n)$ is even if and only if $n \in 2C$,*
- *$H(n)$ is even if and only if $n \in 4C - 1$ or $n \in 4C$, and*
- *$R(n)$ is even if and only if $n \in 2C - 1$.*

Can we obtain and prove such results automatically?

Algebraic sequences

$s(n)_{n\geq 0}$ is algebraic if there is a nonzero polynomial $P(x, y)$ such that

$$P\left(x, \sum_{n\geq 0} s(n)x^n\right) = 0.$$

$C(n)_{n\geq 0}, M(n)_{n\geq 0}, P(n)_{n\geq 0}, H(n)_{n\geq 0}, R(n)_{n\geq 0}$ are all algebraic.

Example

For the Catalan numbers…

$y = \sum_{n\geq 0} C(n)x^n$ satisfies $xy^2 - y + 1 = 0$ over $\mathbb{Q}$.

$y = \sum_{n\geq 0} (C(n) \bmod 3)x^n$ satisfies $xy^2 + 2y + 1 = 0$ over $\mathbb{F}_3$.

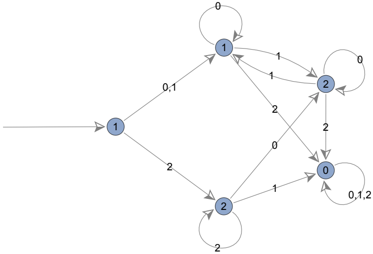
$\mathbb{F}_p$ denotes the finite field with p elements.

Automatic sequences

$s(n)_{n\geq 0}$ is *p-automatic* if there is an automaton that outputs $s(n)$ when fed the base-p digits of n.

Convention in this talk: start with the least significant digit.

This automaton computes $C(n) \bmod 3$:

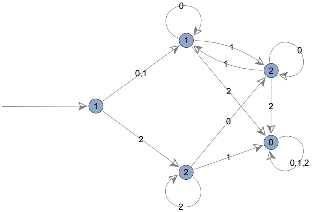


$C(9) \equiv\ ? \mod 3$. Since $9 = 100_3$, $C(9) \equiv \boxed{2} \mod 3$.

$\left(C(n) \bmod 3\right)_{n\geq 0} = 1, 1, 2, 2, 2, 0, 0, 0, 2, 2, \ldots$ is 3-automatic.

Two representations: polynomials and automata.

$$xy^2 + 2y + 1 = 0$$



Polynomial: easy to get from the polynomial over $\mathbb{Q}$.

Automaton: direct information about $s(n)$.

Theorem (Christol 1979/1980)

A sequence $s(n)_{n\geq 0}$ of elements in $\mathbb{F}_p$ is algebraic if and only if it is p-automatic.

How do we convert a polynomial into an automaton?

How does the automaton size depend on the polynomial degree?

How to tell whether a sequence is p-automatic?

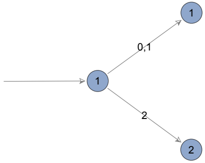
Let $r \in \{0, 1, \ldots, p-1\}$.
The Cartier operator Λ_r picks out every pth term, starting with $s(r)$:

$$\Lambda_r(s(n)_{n\geq 0}) := s(pn+r)_{n\geq 0}$$

Iteratively apply $\Lambda_0, \Lambda_1, \ldots, \Lambda_{p-1}$ to $s(n)_{n\geq 0}$.
Create one state in the automaton for each distinct sequence.

Let $s(n) = (C(n) \bmod 3)$. $\quad s(n)_{n\geq 0} = 1, 1, 2, 2, 2, 0, 0, 0, 2, \ldots$.

$$\begin{aligned}
\Lambda_0(s(n)_{n\geq 0}) &= s(3n+0)_{n\geq 0} = 1, 2, 0, 2, 1, 0, 0, 0, 0, \ldots && \text{new!} \\
\Lambda_1(s(n)_{n\geq 0}) &= s(3n+1)_{n\geq 0} = 1, 2, 0, 2, 1, 0, 0, 0, 0, \ldots && = \Lambda_0(s(n)_{n\geq 0}) \\
\Lambda_2(s(n)_{n\geq 0}) &= s(3n+2)_{n\geq 0} = 2, 0, 2, 1, 0, 0, 0, 0, 2, \ldots && \text{new!}
\end{aligned}$$



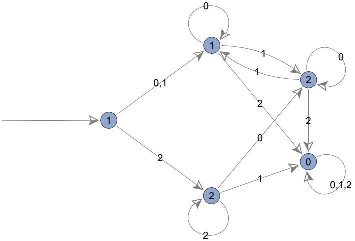
Label each state with the initial term of the corresponding sequence.

$\Lambda_0(\Lambda_0(s(n)_{n\geq 0})) = 1, 2, 0, 2, 1, 0, 0, 0, 0, 2, \ldots = \Lambda_0(s(n)_{n\geq 0})$

$\Lambda_1(\Lambda_0(s(n)_{n\geq 0})) = 2, 1, 0, 1, 2, 0, 0, 0, 0, 1, \ldots$ new!

$\Lambda_2(\Lambda_0(s(n)_{n\geq 0})) = 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots$ new!

$\Lambda_r(\Lambda_2(s(n)_{n\geq 0})) \quad \ldots$



Eilenberg 1974:
A sequence is *p*-automatic if and only if this process terminates.

But we can’t tell if sequences are equal from finitely many terms!

Use a different representation: diagonals of rational functions.

$\sum_{n\geq 1} (C(n) \bmod 3)x^n$ is the diagonal of

$$\begin{aligned}
\frac{y\frac{\partial P}{\partial y}(xy,y)}{P(xy,y)/y} = \frac{y(2xy^2+(2xy+2))}{xy^2+(2xy+2)+x} = \quad & 0x^0y^0 + 1x^0y^1 + 0x^0y^2 + 0x^0y^3 + 0x^0y^4 + 0x^0y^5 + \cdots \\
& + 0x^1y^0 + 1x^1y^1 + 0x^1y^2 + 2x^1y^3 + 0x^1y^4 + 0x^1y^5 + \cdots \\
& + 0x^2y^0 + 1x^2y^1 + 2x^2y^2 + 0x^2y^3 + 1x^2y^4 + 2x^2y^5 + \cdots \\
& + 0x^3y^0 + 1x^3y^1 + 1x^3y^2 + 2x^3y^3 + 0x^3y^4 + 1x^3y^5 + \cdots \\
& + 0x^4y^0 + 1x^4y^1 + 0x^4y^2 + 2x^4y^3 + 2x^4y^4 + 0x^4y^5 + \cdots \\
& + 0x^5y^0 + 1x^5y^1 + 2x^5y^2 + 0x^5y^3 + 0x^5y^4 + 0x^5y^5 + \cdots \\
& + \cdots .
\end{aligned}$$

Theorem (Furstenberg 1967)

Let K be a field, and let $P(x,y) \in K[x,y]$ such that $\frac{\partial P}{\partial y}(0,0) \neq 0$. If $F(x) \in K[\![x]\!]$ satisfies $F(0) = 0$ and $P(x, F(x)) = 0$, then

$$F(x) = \operatorname{diag}\left(\frac{y\frac{\partial P}{\partial y}(xy,y)}{P(xy,y)/y}\right).$$

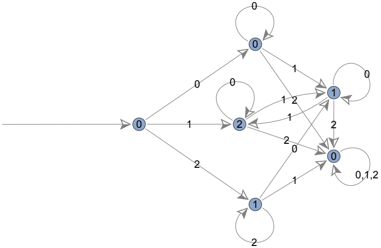
We have embedded $s(n)_{n\geq 1}$ into a series $\frac{S_0}{Q} := \frac{y(2xy^2+(2xy+2))}{xy^2+(2xy+2)+x}$.

Construct an automaton by iterating $\lambda_{r,r}(S) := \Lambda_{r,r}\left(S \cdot Q^{p-1}\right)$.

$$\lambda_{0,0}(S_0) = xy^2 + xy \qquad \text{new!}$$
$$\lambda_{1,1}(S_0) = 2 \qquad \text{new!}$$
$$\lambda_{2,2}(S_0) = y + 1 \qquad \text{new!}$$

$$\lambda_{0,0}(xy^2 + xy) = xy^2 + xy = \lambda_{0,0}(S_0) \qquad \ldots$$

Create one state in the automaton for each distinct polynomial.



The automaton may not be minimal.

Prime power moduli

This algorithm can be adapted to work modulo p^α.

Theorem (Denef–Lipshitz 1987)

Let $\alpha \geq 1$. *Let* $R(\mathbf{x}), Q(\mathbf{x}) \in \mathbb{Z}_p[\mathbf{x}]$ *such that* $Q(0, \ldots, 0) \not\equiv 0 \mod p$. *Then the coefficient sequence of* $\left(\operatorname{diag} \frac{R(\mathbf{x})}{Q(\mathbf{x})}\right) \mod p^\alpha$ *is p-automatic.*

$\mathbb{Z}_p$ denotes the set of p-adic integers.

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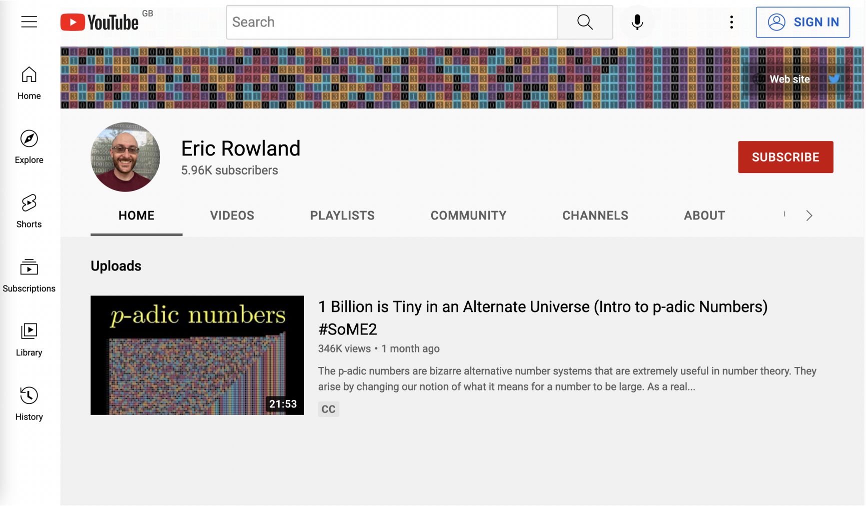

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The p-adic numbers are bizarre alternative number systems that are extremely useful in number theory. They arise by changing our notion of what it means for a number to be large. As a real...

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Prime power moduli

This algorithm can be adapted to work modulo p^α.

Theorem (Denef–Lipshitz 1987)

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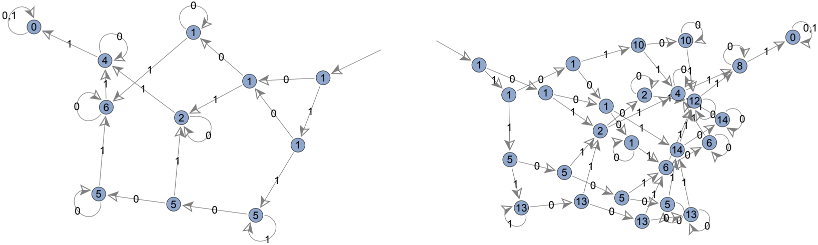
$\mathbb{Z}_p$ denotes the set of p-adic integers.

By computing an automaton for a sequence $\bmod p^\alpha$, we can answer…

- Are there forbidden residues?
- What is the limiting distribution of residues (if it exists)?
- Is the sequence eventually periodic?
- Many other questions known to be decidable.

Catalan numbers modulo 8 and modulo 16:



Theorem (Liu–Yeh)

$C(n) \not\equiv 9 \mod 16$ *for all* $n \geq 0$.

Proof: Compute the automaton.

Catalan numbers modulo 2^α:

Theorem (Rowland–Yassawi 2015)

For all $n \geq 0$,

- $C(n) \not\equiv 17, 21, 26 \mod 32$,
- $C(n) \not\equiv 10, 13, 33, 37 \mod 64$,
- $C(n) \not\equiv 18, 54, 61, 65, 66, 69, 98, 106, 109 \mod 128$.

Only $\approx$ 35% of the residues modulo 512 are attained by some $C(n)$.

Open question

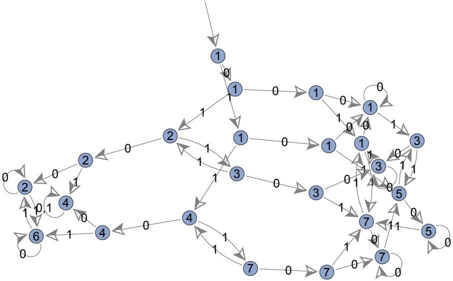
Does the density of residues modulo 2^α that are attained by some Catalan number tend to 0 as α gets large?

For the Motzkin numbers…

Theorem (Eu–Liu–Yeh; conj. by Deutsch–Sagan–Amdeberhan)

$M(n) \not\equiv 0 \mod 8$ for all $n \geq 0$.

Proof: $M(n) \bmod 8$ is computed by the following automaton.



Theorem (Rowland–Yassawi)

$M(n) \not\equiv 0 \mod 5^2$ and $M(n) \not\equiv 0 \mod 13^2$ for all $n \geq 0$.

Apéry numbers

$A(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ arose in Apéry's proof that $\zeta(3)$ is irrational.

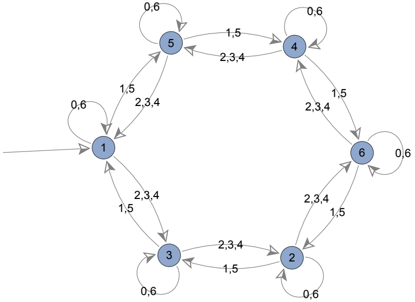
$A(n)_{n \geq 0} = 1, 5, 73, 1445, 33001, 819005, 21460825, \dots$

Straub 2014: $\sum_{n \geq 0} A(n) x^n$ is the diagonal of

$$\frac{1}{(1-w-x)(1-y-z)-wxyz}.$$

Therefore $(A(n) \bmod p^\alpha)_{n \geq 0}$ is p-automatic.

$A(n)$ modulo 7: $1, 5, 3, 3, 3, 5, 1, 5, 4, 1, 1, \ldots$



Theorem (Gessel 1982)

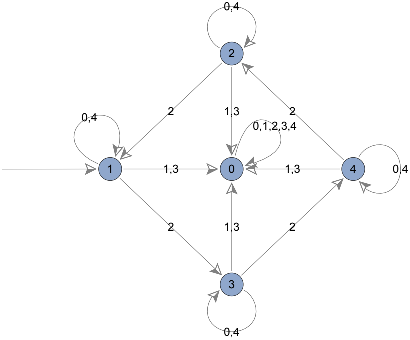
Let p be a prime. The Apéry numbers satisfy the Lucas congruence

$$A(pn + d) \equiv A(n)A(d) \mod p$$

for all $n \geq 0$ *and all* $d \in \{0, 1, \ldots, p - 1\}$.

$$A(2039) = A(5642_7) \equiv A(5)A(6)A(4)A(2) \equiv 5 \cdot 1 \cdot 3 \cdot 3 \equiv 3 \mod 7$$

$A(n)$ modulo 5: $1, 0, 3, 0, 1, 0, 0, 0, 0, 0, 3, \ldots$



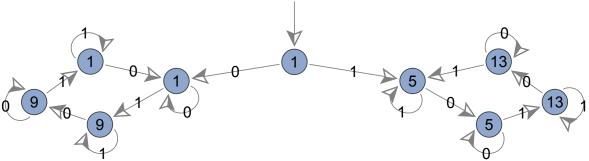
If the base-5 representation of n contains 1 or 3, then $A(n) \equiv 0 \mod 5$.

$A(n)$ modulo 2^α…

Gessel proved the conjecture of Chowla–Cowles–Cowles (1980) that

$$A(n) \bmod 8 = \begin{cases} 1 & \text{if } n \text{ is even} \\ 5 & \text{if } n \text{ is odd.} \end{cases}$$

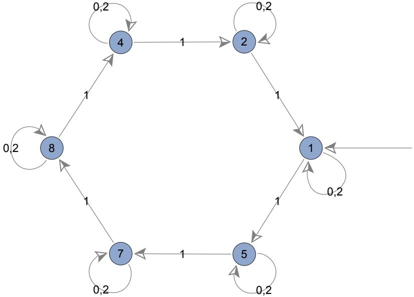
Gessel asked whether $A(n)$ is periodic modulo 16.



Theorem (Rowland–Yassawi)

$(A(n) \bmod 16)_{n \geq 0}$ *is not eventually periodic.*

$A(n)$ modulo 9:

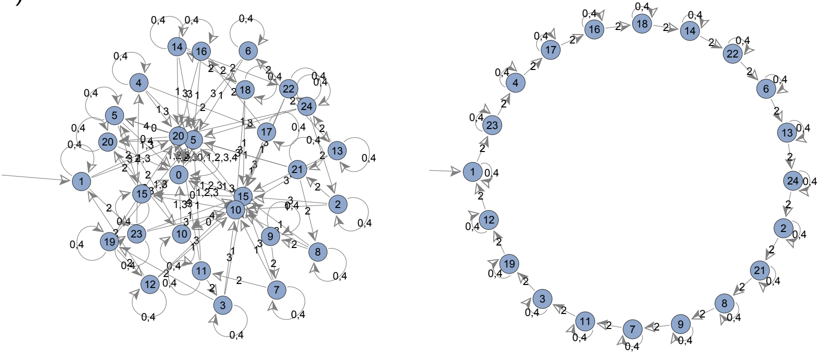


Theorem (Gessel)

$A(3n+d) \equiv A(n)A(d) \mod 9$ *for all* $n \geq 0$ *and all* $d \in \{0,1,2\}$.

For $p \geq 5$, the Lucas congruence does not always hold modulo p^2.

$A(n)$ modulo 25:



Restrict the digit set.

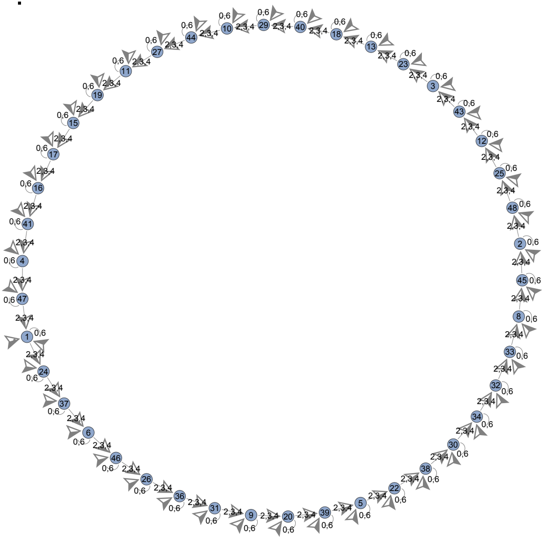
$A(2) = 73 \equiv -2 \mod 25$

Theorem (Rowland–Yassawi)

$A(5n + d) \equiv A(n)A(d) \mod 25$ *for all* $n \geq 0$ *and all* $d \in \{0, 2, 4\}$*.*

Which digits support a Lucas congruence for $A(n)$ modulo p^2?

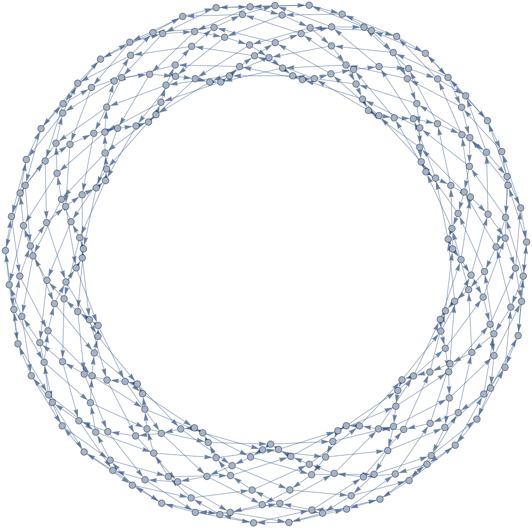
$A(n)$ modulo 7^2:



digit set: $\{0, 2, 3, 4, 6\}$

$(A(0), A(2), A(3), A(4), A(6)) \equiv (1, 24, 24, 24, 1) \mod 7^2$

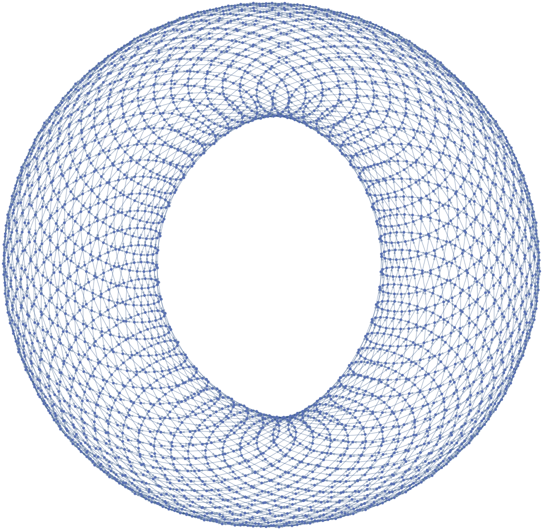
$A(n)$ modulo 23^2:



digit set: $\{0, 7, 11, 15, 22\}$

$(A(0), A(7), A(11), A(15), A(22)) \equiv (1, 415, 473, 415, 1) \mod 23^2$

$A(n)$ modulo 59^2:



digit set: $\{0, 6, 29, 52, 58\}$

$(A(0), A(6), A(29), A(52), A(58)) \equiv (1, 460, 2813, 460, 1) \mod 59^2$

Theorem (Malik–Straub 2016)

$A(d) \equiv A(p-1-d) \mod p$ for each $d \in \{0, 1, \dots, p-1\}$.

$$(A(0), A(1), \dots, A(10)) \equiv (1, 5, 7, 4, 1, 0, 1, 4, 7, 5, 1) \mod 11$$

Let $D(p) := \left\{ d \in \{0, 1, \dots, p-1\} : A(d) \equiv A(p-1-d) \mod p^2 \right\}$.

In particular, $\{0, \frac{p-1}{2}, p-1\} \subseteq D(p)$. $\{0, 2, 4\} \subseteq D(5)$

Theorem (Rowland–Yassawi–Krattenthaler 2021)

Let p be a prime and $d \in \{0, 1, \dots, p-1\}$. The congruence

$$A(pn + d) \equiv A(n)A(d) \mod p^2$$

holds for all $n \geq 0$ if and only if $d \in D(p)$.

Primes p with $|D(p)| \geq 4$:

p	$D(p)$
7	$\{0, 2, 3, 4, 6\}$
23	$\{0, 7, 11, 15, 22\}$
43	$\{0, 5, 18, 21, 24, 37, 42\}$
59	$\{0, 6, 29, 52, 58\}$
79	$\{0, 18, 39, 60, 78\}$
103	$\{0, 17, 51, 85, 102\}$
107	$\{0, 14, 21, 47, 53, 59, 85, 92, 106\}$
127	$\{0, 17, 63, 109, 126\}$
131	$\{0, 62, 65, 68, 130\}$
139	$\{0, 68, 69, 70, 138\}$
151	$\{0, 19, 75, 131, 150\}$
167	$\{0, 35, 64, 83, 102, 131, 166\}$

Heuristic probability that $|D(p)| \geq 4$:

$$1 - \left(1 - \frac{1}{p}\right)^{(p-3)/2} \quad \to \quad 1 - \frac{1}{\sqrt{e}} \approx .393$$

How does the size of the automaton (number of states) depend on the x-degree (height) and y-degree (degree) of the polynomial?

Theorem (Bridy 2017)

Let $s(n)_{n \geq 0}$ be an algebraic sequence of elements in $\mathbb{F}_p$. If its minimal polynomial has height h, degree d, and genus g, then the number of states in its minimal automaton is at most

$$(1 + o(1))p^{h+d+g-1},$$

where $o(1)$ tends to 0 as any of p, h, d, g gets large.

The genus satisfies $g \leq (h-1)(d-1)$; generically $g = (h-1)(d-1)$.

Corollary

The number of states is at most $(1 + o(1))p^{hd}$.

Can we get this bound without algebraic geometry? Yes.

Is the bound sharp? We suspect yes.

Corollary

The number of states is at most $(1 + o(1))p^{hd}$.

The factor $1 + o(1)$ cannot be removed.

Example

Let $p = 2$ and

$$P = (x^3 + x^2 + 1)y^3 + (x^3 + 1)y^2 + (x^3 + x^2 + x + 1)y + x^3 + x^2$$

with $h = 3$ and $d = 3$. The number of states is $532 > 512 = p^{hd}$.