Unanswered questions about the Fibonacci numbers

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Fibonacci sequence

Define $F(0) = 0$, $F(1) = 1$, and for $n \geq 2$

$$F(n) = F(n-1) + F(n-2).$$

$F(n)_{n \geq 0}$: $\quad 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$

Rabbits:
Start with a pair of rabbits. Each pair of rabbits produces one new pair each month. New rabbits take one month to mature.

Stairs:
How many ways are there to climb n stairs, climbing 1 or 2 each step?

$n = 4$:	1111	112	121	211	22			
$n = 5$:	11111	1121	1211	2111	221	1112	122	212

Experiment 1

What does the Fibonacci sequence look like modulo m?

- $(F(n) \bmod m)_{n \geq 0}$ is periodic.
- If m factors as $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, then the period length is

$$\ell(m) = \operatorname{lcm}\big(\ell(p_1^{\alpha_1}), \dots, \ell(p_k^{\alpha_k})\big) .$$

Period length for prime powers

Theorem (D. D. Wall 1960)

If $\ell(p^2) \neq \ell(p)$, then $\ell(p^\alpha) = p^{\alpha-1}\ell(p)$.

"The most perplexing problem we have met in this study concerns the hypothesis $\ell(p^2) \neq \ell(p)$. We have run a test on a digital computer which shows that $\ell(p^2) \neq \ell(p)$ for all p up to $10{,}000$; however, we cannot yet prove that $\ell(p^2) = \ell(p)$ is impossible."

A prime satisfying $\ell(p^2) = \ell(p)$ is called a Wall–Sun–Sun prime.

Connection to Fermat's last theorem:

Theorem (Zhi-Hong Sun and Zhi-Wei Sun 1992)

If $p \neq 2$, $p \nmid xyz$, and $x^p + y^p = z^p$, then $\ell(p^2) = \ell(p)$.

To be or not to be?

http://www.primegrid.com:
There is no Wall–Sun–Sun prime less than 1.9×10^{17}.

Unanswered question

Are there any Wall–Sun–Sun primes?

Equivalent characterization: $p^2 \mid F(p \pm 1)$, where

$$\pm 1 = \begin{cases} -1 & \text{if } p \equiv 1 \text{ or } 4 \mod 5 \\ +1 & \text{if } p \equiv 2 \text{ or } 3 \mod 5. \end{cases}$$

If the p^1 digit of $F(p \pm 1)$ is random, then one expects the number of Wall–Sun–Sun primes $\leq x$ to be

$$\sum_{p \leq x} \frac{1}{p} \approx \log \log_2 x.$$

We should have already found $\log \log_2(1.9 \times 10^{17}) = 4.05$.

Experiment 2

How many residues are attained by $F(n)_{n \geq 0}$ modulo p^α?

Unanswered question

What is the limiting density

$$\lim_{\alpha \to \infty} \frac{|\{F(n) \bmod p^\alpha : n \geq 0\}|}{p^\alpha}$$

of residues attained by the Fibonacci sequence modulo p^α?

Conjecture:
For $p = 2$ the limiting density is $\frac{21}{32}$.

Known densities

Burr (1971) completed the characterization of integers m such that $(F(n) \bmod m)_{n \geq 0}$ attains all residues modulo m:

$$2 \cdot 5^\alpha, \quad 4 \cdot 5^\alpha, \quad 6 \cdot 5^\alpha, \quad 7 \cdot 5^\alpha, \quad 14 \cdot 5^\alpha, \quad 3^\beta \cdot 5^\alpha$$

for all $\alpha \geq 0$ and $\beta \geq 0$.

In particular, for $p = 3$ and $p = 5$ the limiting density is

$$\lim_{\alpha \to \infty} \frac{|\{F(n) \bmod p^\alpha : n \geq 0\}|}{p^\alpha} = 1.$$

Theorem (Rowland–Yassawi 2016)

For $p = 11$ the limiting density is $\frac{145}{264}$.

Experiment 3

Is there structure in the base-2 representations of Fibonacci numbers?

$\lim_{n\to\infty} F(2^{2n})$ and $\lim_{n\to\infty} F(2^{2n+1})$ are equal to $\pm\sqrt{-\frac{3}{5}}$.

$$1 + 0\cdot 2^1 + 0\cdot 2^2 + 1\cdot 2^3 + 1\cdot 2^4 + 0\cdot 2^5 + 0\cdot 2^6 + 1\cdot 2^7 + \cdots$$
$$= \sum_{n\geq 0}\left(2^{4n} + 2^{4n+3}\right) = \frac{1}{1-2^4} + \frac{2^3}{1-2^4} = -\frac{3}{5}$$

We can justify use of the geometric series formula if 2 is "small".

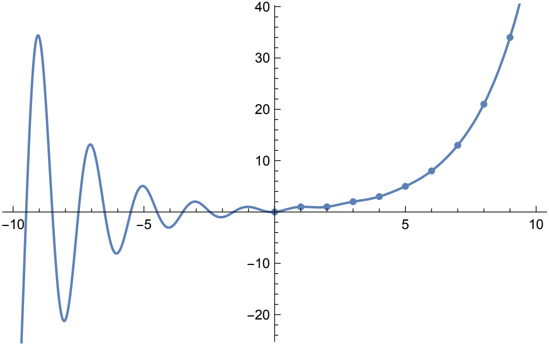
The 2-adic absolute value $|\cdot|_2$ is defined on $\mathbb{Q}$ by $|0|_2 := 0$ and $|\frac{a}{b}2^n|_2 := \frac{1}{2^n}$ for odd a, b and $n \in \mathbb{Z}$.

The set of 2-adic integers is denoted $\mathbb{Z}_2$.

Interpolation to $\mathbb{R}$

Let $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$. Binet's formula:

$$F(n) = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}$$



$$F(x) = \frac{\exp(x \log \phi) - \cos(\pi x) \exp(x \log(-\bar{\phi}))}{\sqrt{5}}$$

Twisted interpolation to $\mathbb{Z}_{11}$

$F(n)$ cannot be interpolated to $\mathbb{Z}_{11}$.
But it can be interpolated on each residue class modulo 10.

Let $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$ in $\mathbb{Z}_{11}$.
Let $\omega(\phi), \omega(\bar{\phi}) \in \mathbb{Z}_{11}$ be 10th roots of unity congruent to $\phi, \bar{\phi}$ mod 11.

Theorem (Rowland–Yassawi 2016)

For each $0 \leq i \leq 9$, the function $F_i : \mathbb{Z}_{11} \to \mathbb{Z}_{11}$ defined by

$$F_i(x) := \frac{\omega(\phi)^i \exp_{11}\left(x \log_{11} \frac{\phi}{\omega(\phi)}\right) - \omega(\bar{\phi})^i \exp_{11}\left(x \log_{11} \frac{\bar{\phi}}{\omega(\bar{\phi})}\right)}{\sqrt{5}}$$

is the unique continuous function such that $F(n) = F_i(n)$ for all $n \equiv i \mod 10$.

Fibonacci residues modulo 11^α

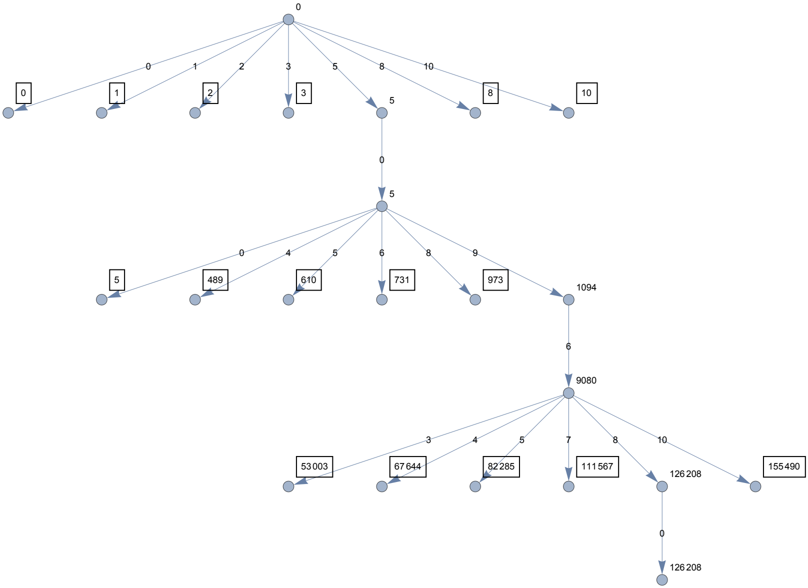
Let μ be the Haar measure on $\mathbb{Z}_p$ defined by $\mu(m + p^\alpha \mathbb{Z}_p) = \frac{1}{p^\alpha}$.

Theorem (Rowland–Yassawi 2016)

The limiting density of residues attained by the Fibonacci sequence modulo 11^α *is*

$$\lim_{\alpha\to\infty} \frac{|\{F(n) \bmod 11^\alpha : n \geq 0\}|}{11^\alpha} = \mu\left(\bigcup_{i=0}^{9} F_i(\mathbb{Z}_{11})\right) = \frac{145}{264}.$$

Fibonacci residues modulo 11^α



Fibonacci residues modulo p^α

If $p \equiv 1$ or $4 \mod 5$, then $x^2 = 5$ has a solution in $\mathbb{Z}_p$.

If $p \equiv 2$ or $3 \mod 5$, then $x^2 = 5$ does not have a solution in $\mathbb{Z}_p$.
We need to work in the extension $\mathbb{Q}_p(\sqrt{5})$.

(If $p = 5$, we already know the limiting density of residues is 1.)

More general sequences

Let $s(n)_{n\geq 0}$ be a sequence of p-adic integers satisfying a recurrence

$$s(n+d) + a_{d-1}s(n+d-1) + \cdots + a_1 s(n+1) + a_0 s(n) = 0$$

with constant coefficients $a_j \in \mathbb{Z}_p$.

In general $s(n)$ cannot be interpolated to $\mathbb{Z}_p$.
But it can be approximated by a continuous function $s_i(x)$ on each residue class modulo m.

The limiting density of attained residues is

$$\lim_{\alpha\to\infty} \frac{|\{s(n) \bmod p^\alpha : n \geq 0\}|}{p^\alpha} = \mu\left(\mathbb{Z}_p \cap \bigcup_i s_i(r + p^e \mathbb{Z}_p)\right).$$