Two simple questions without simple answers

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joint work with August Fogler and Aidan Hackett

Mathematics Seminar
Hofstra University, 2025–09–17

Project 1

1000 contains 00.
0110 avoids 00.

How many length-n binary words avoid 00?

length 0:	empty word	1
length 1:	$0, 1$	2
length 2:	$01, 10, 11$	3
length 3:	$010, 011, 101, 110, 111$	5
length 4:	$0101, 0110, 0111, 1010, 1011, 1101, 1110, 1111$	8
$\vdots$		$\vdots$
length n:		$F(n+2)$

Fibonacci recurrence: $F(n) = F(n-1) + F(n-2)$

Question

How many $m \times n$ binary arrays avoid $\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$?

$\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}$ avoids $\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$.

Fix $m = 2$ and vary n...

$n = 0$:	empty array	1
$n = 1$:	$\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}, \begin{smallmatrix} 0 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 0 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$	4
$n = 2$:	$\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}, \dots, \begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$	15
$n = 3$:	$\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}, \dots, \begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}$	57
$n = 4$:	$\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{smallmatrix}, \begin{smallmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{smallmatrix}, \dots, \begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}$	216

Recurrence: $a(n) = 3a(n-1) + 3a(n-2)$

THE ON–LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

A181253 T(n,k)=Number of nXk binary matrices with no 2X2 block having four 1's. 8

2, 4, 4, 8, 15, 8, 16, 57, 57, 16, 32, 216, 417, 216, 32, 64, 819, 3032, 3032, 819, 64, 128, 3105, 22077, 42176, 22077, 3105, 128, 256, 11772, 160697, 587920, 587920, 160697, 11772, 256, 512, 44631, 1169792, 8191392, 15701273, 8191392, 1169792, 44631, 512

(list; table; graph; refs; listen; history; text; internal format)

OFFSET 1,1

LINKS R. H. Hardin, Table of n, a(n) for n=1..721

FORMULA
Empirical column 1: a(n)=2*a(n-1)
Empirical column 2: a(n)=3*a(n-1)+3*a(n-2)
Empirical column 3: a(n)=6*a(n-1)+10*a(n-2)-5*a(n-3)
Empirical column 4: a(n)=10*a(n-1)+54*a(n-2)+16*a(n-3)-64*a(n-4)
Empirical column 5: a(n)=20*a(n-1)+188*a(n-2)-192*a(n-3)-1660*a(n-4)+2804*a(n-5)-507*a(n-6)-624*a(n-7)
Empirical column 6: a(n)=33*a(n-1)+908*a(n-2)+1687*a(n-3)-37947*a(n-4)-16572*a(n-5)+513993*a(n-6)-663729*a(n-7)-486540*a(n-8)+617409*a(n-9)+191835*a(n-10)-49140*a(n-11)
Empirical column 7: a(n)=68*a(n-1)+3106*a(n-2)-10300*a(n-3)-731184*a(n-4)+3930848*a(n-5)+47046600*a(n-6)-471525808*a(n-7)+1012118640*a(n-8)+2396096576*a(n-9)-9445394304*a(n-10)-4382776896*a(n-11)+29415041536*a(n-12)+8676097024*a(n-13)-36065068032*a(n-14)-14871987200*a(n-15)+10138337280*a(n-16)+2907136000*a(n-17)-1119682560*a(n-18)
Empirical column 8: a(n)=113*a(n-1)+13879*a(n-2)+91506*a(n-3)-13567062*a(n-4)-45766270*a(n-5)+5948333641*a(n-6)-25692714697*a(n-7)-932093986319*a(n-8)+9749317949468*a(n-9)+6293344318720*a(n-10)-400364584466276*a(n-11)+544975615003201*a(n-12)+8011657063605359*a(n-13)-12237642139437047*a(n-14)-98976024373360414*a(n-15)+87321080164809042*a(n-16)+743714645681446194*a(n-17)-219417428841728730*a(n-18)-2838216189512832023*a(n-19)-1559534908222727729*a(n-20)+4451110188283146640*a(n-21)+3110756142589939204*a(n-22)-3806251587192837456*a(n-23)-2258950594106495040*a(n-24)+1998716044109621760*a(n-25)+565195437997056000*a(n-26)-541032812384256000*a(n-27)+28184753405952000*a(n-28)+1949377757184000 0*a(n-29)

EXAMPLE
Table starts
```
....2......4.........8...........16..............32..................64
....4.....15........57..........216.............819................3105
....8.....57.......417.........3032...........22077..............160697
...16....216......3032........42176..........587920.............8191392
...32....819.....22077.......587920........15701273...........419045269
...64...3105....160697......8191392.......419045269.........21418970801
..128..11772...1169792....114142368.....11185495872.......1095020802848
..256..44631...8515337...1590466304....298561305103.....55979092539545
..512.169209..61986457..22161786304...7969215344753...2861765993703849
.1024.641520.451223152.308805072256.212714316418464.146298965997241152
```

CROSSREFS
Diagonal is A139810.
Column 2 is A125145.
Sequence in context: A282528 A297094 A283282 * A267788 A189696 A189196
Adjacent sequences: A181250 A181251 A181252 * A181254 A181255 A181256

KEYWORD nonn,tabl

AUTHOR R. H. Hardin, Oct 10 2010

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  18)-2838216189512832023*a(n-19)-1559534908222727729*a(n-20)+4451110188283146640*a(n-
  21)+3110756142589939204*a(n-22)-3806251587192837456*a(n-23)-2258950594106495040*a(n-
  24)+1998716044109621760*a(n-25)+565195437997056000*a(n-26)-541032812384256000*a(n-
  27)+28184753405952000*a(n-28)+19493777571840000*a(n-29)
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Conjecture

The size $r(m)$ *of the recurrence for* $m \times n$ *arrays avoiding* $\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$ *satisfies*

$$r(m) = r(m-1) + r(m-2)$$

for all $m \geq 5$.

Project 2

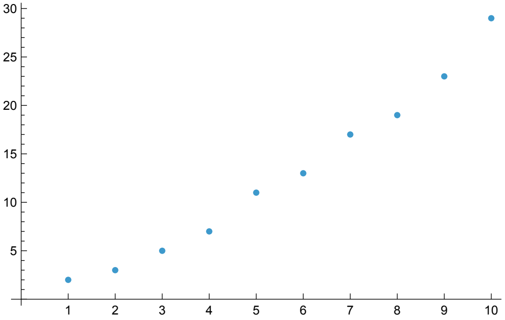
$2, 3, 5, 7, 11, \ldots$

How big is the *n*th prime?

Project 2

$2, 3, 5, 7, 11, \ldots$

How big is the *n*th prime?



Project 2

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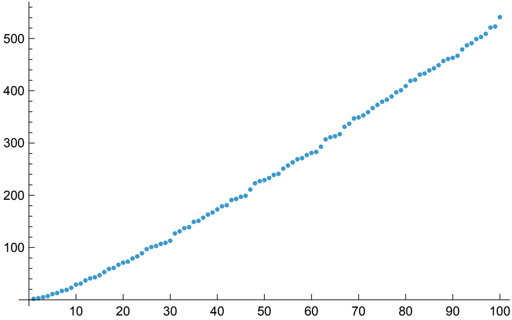

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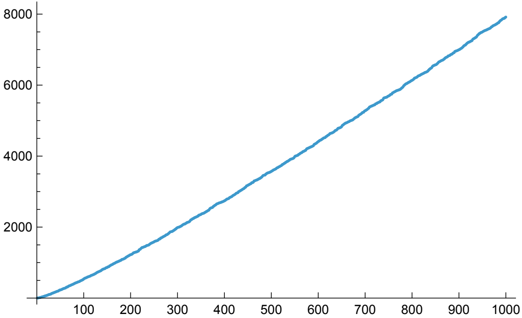

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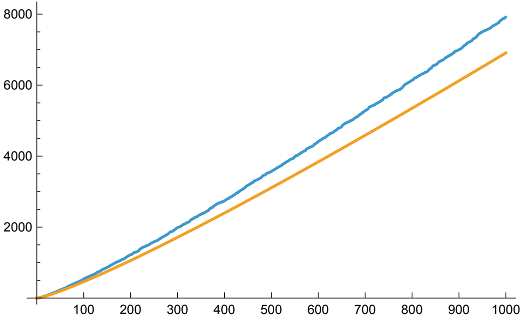

Prime number theorem (Hadamard and de la Vallée Poussin, 1896)

The nth prime is asymptotically $n \log n$*.*

Chebyshev, 1850:

$$\frac{(n)!(30n)!}{(6n)!(10n)!(15n)!}$$

is an integer for each $n \geq 0$.

$n = 0$: 1
$n = 1$: 77636318760
$n = 2$: 53837289804317953893960
$n = 3$: 43880754270176401422739454033276880
$n = 4$: 38113558705192522309151157825210540422513019720

Balanced:
$1 + 30 = 6 + 10 + 15$; one more factorial in denominator than numerator

Theorem (Rodriguez Villegas, 2007)

Let $s(n)$ be a balanced factorial ratio.
Then $s(n)$ is an integer for each $n \geq 0$ if and only if $\sum_{n \geq 0} s(n)x^n$ is algebraic.

Examples of algebraic series:

$$y = \sum_{n \geq 0} x^n = 1 + x + x^2 + \cdots = \frac{1}{1-x} \quad \text{satisfies } 1 - y + xy = 0.$$

$$y = \sum_{n \geq 0} \frac{(2n)!}{n!^2} x^n = 1 + 2x + 6x^2 + \cdots \quad \text{satisfies } 1 - y^2 + 4xy^2 = 0.$$

$$y = \sum_{n \geq 0} \frac{(n)!(30n)!}{(6n)!(10n)!(15n)!} x^n \quad \text{satisfies a degree-483840 equation!}$$

Question

How does the degree depend on the coefficients in the factorial ratio?

Simple family of balanced factorial ratios:

$$\binom{an}{bn} = \frac{(an)!}{(bn)!((a-b)n)!}$$

Balanced:
$a = b + (a - b)$; one more factorial in denominator than numerator

Let $b = 1$.

$$y = \sum_{n \geq 0} \binom{n}{n} x^n \qquad 1 - y + xy = 0$$

$$y = \sum_{n \geq 0} \binom{2n}{n} x^n \qquad 1 - y^2 + 4xy^2 = 0$$

$$y = \sum_{n \geq 0} \binom{3n}{n} x^n \qquad 1 + 3y - 4y^3 + 27xy^3 = 0$$

$$y = \sum_{n \geq 0} \binom{4n}{n} x^n \qquad 1 + 8y + 18y^2 - 27y^4 + 256xy^4 = 0$$

$a = 1$					1	-1		1
$a = 2$				1	0	-1		4
$a = 3$			1	3	0	-4		27
$a = 4$		1	8	18	0	-27		256
$a = 5$	1	15	80	160	0	-256		3125
$a = 6$	1	24	225	1000	1875	0	-3125	46656
	y^0	y^1	y^2	y^3	y^4	y^5	y^6	xy^a

1

	y^0	y^1	y^2	y^3	y^4	y^5	y^6	xy^a
$a = 1$	1	-1						1
$a = 2$	1	0	-1					4
$a = 3$	1	3	0	-4				27
$a = 4$	1	8	18	0	-27			256
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$a=1$	1	−1						1
$a=2$	1	0	−1					4
$a=3$	1	3	0	−4				27
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1

$a(a-2)$

	y^0	y^1	y^2	y^3	y^4	y^5	y^6	xy^a
$a = 1$	1	-1						1
$a = 2$	1	0	-1					4
$a = 3$	1	3	0	-4				27
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1

$a(a-2)$

$\frac{1}{2}a(a-1)^2(a-3)$

	y^0	y^1	y^2	y^3	y^4	y^5	y^6	xy^a	
$a = 1$				1	−1			1	1
$a = 2$			1	0	−1			4	$a(a-2)$
$a = 3$		1	3	0	−4			27	$\frac{1}{2}a(a-1)^2(a-3)$
$a = 4$	1	8	18	0	−27			256	$\frac{1}{6}a(a-1)^3(a-2)(a-4)$
$a = 5$	1	15	80	160	0	−256		3125	
$a = 6$	1	24	225	1000	1875	0	−3125	46656	

	y^0	y^1	y^2	y^3	y^4	y^5	y^6	xy^a	
$a = 1$	1	-1						1	1
$a = 2$	1	0	-1					4	$a(a-2)$
$a = 3$	1	3	0	-4				27	$\frac{1}{2}a(a-1)^2(a-3)$
$a = 4$	1	8	18	0	-27			256	$\frac{1}{6}a(a-1)^3(a-2)(a-4)$
$a = 5$	1	15	80	160	0	-256		3125	$\frac{1}{24}a(a-1)^4(a-2)(a-3)(a-5)$
$a = 6$	1	24	225	1000	1875	0	-3125	46656	

Conjecture

If $a \geq 2$, then the series $y = \sum_{n \geq 0} \binom{an}{n} x^n$ satisfies

$$\sum_{i=0}^{a} (a-1)^{i-1}(a-i-1)\binom{a}{i} y^i + a^a x y^a = 0.$$

Degree as we vary a, b:

1							
1	1						
1	2	1					
1	3	3	1				
1	4	4	4	1			
1	5	10	10	5	1		
1	6	9	8	9	6	1	
1	7	21	35	35	21	7	1

Conjecture

The algebraic degree of the series $\sum_{n\geq 0} \binom{an}{bn} x^n$ *is at most* $\binom{a}{b}$.