

Two binomial coefficient conjectures

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Outline

- 1 Counting entries modulo m
- 2 The value of $\binom{n}{m}$ modulo n

First question

Arithmetic properties of binomial coefficients have a long history, the theorems of Kummer and Lucas being classical results.

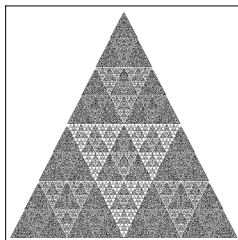
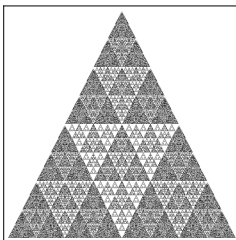
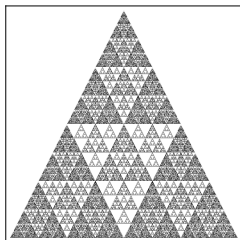
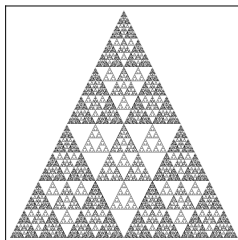
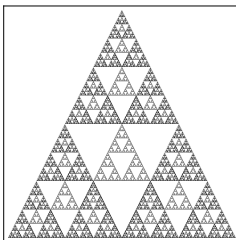
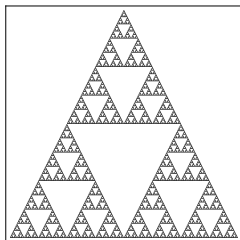
Main theme:

Properties of $\binom{n}{m}$ modulo p are related to the base- p representations $n_l n_{l-1} \cdots n_1 n_0$ and $m_l m_{l-1} \cdots m_1 m_0$.

Let $a_{k,r}(n)$ be the number of $0 \leq m \leq n$ such that $\binom{n}{m} \equiv r \pmod{k}$.

What is the structure of $a_{k,r}(n)$?

Binomial coefficients modulo 2^α



Let $|n|_w$ be the number of occurrences of the word w in $n_l n_{l-1} \cdots n_1 n_0$.

- Glaisher, 1899:

$$a_{2,1}(n) = 2^{|n|_1}$$

- Hexel and Sachs, 1978:

- formula for $a_{p,r^i}(n)$ in terms of $(p-1)$ th roots of unity
- $a_{3,1}(n) = 2^{|n|_1-1} \cdot (3^{|n|_2} + 1)$ and $a_{3,2}(n) = 2^{|n|_1-1} \cdot (3^{|n|_2} - 1)$
- explicit formulas for $a_{5,r^i}(n)$ in terms of $|n|_1$, $|n|_2$, $|n|_3$, and $|n|_4$

- Garfield and Wilf, 1992:

method to compute the generating function $\sum_{i=0}^{p-2} a_{p,r^i}(n)x^i$

- Amdeberhan and Stanley:

Let $f(\mathbf{x}) \in \mathbb{F}_q[x_1, \dots, x_m]$. The number of coefficients in $f(\mathbf{x})^n$ equal to $r \in \mathbb{F}_q^\times$ is a q -regular sequence.

Davis and Webb, 1991:

- $a_{4,1}(n) = \begin{cases} 2^{|n|_1} & \text{if } |n|_{11} = 0 \\ 2^{|n|_1-1} & \text{otherwise} \end{cases}$
- $a_{4,2}(n) = 2^{|n|_1-1} \cdot |n|_{10}$
- $a_{4,3}(n) = \begin{cases} 0 & \text{if } |n|_{11} = 0 \\ 2^{|n|_1-1} & \text{otherwise} \end{cases}$

(Here $|n|_w$ counts occurrences in the base-2 representation of n .)

In particular, if r is odd then $a_{4,r}(n)$ is either 0 or a power of 2.

By Glaisher's result, $a_{2,1}(n) = 2^{|n|_1}$ is (either 0 or) a power of 2.

“Zaphod Beeblebrox's brain and the fifty-ninth row of Pascal's triangle”

Theorem (Granville, 1992)

If r is odd then $a_{8,r}(n)$ is either 0 or a power of 2.

So what about modulo 16?

The statement fails for $n = 59$!

“Unbelievably, there are exactly six entries of Row 59 in each of the congruence classes 1, 11, 13, and 15 (mod 16)! Our pattern has come to an end, but not before providing us with some interesting mathematics, as well as a couple of pleasant surprises.”

This may not be the end of the story ...

What values does $a_{16,r}(n)$ take for odd r ? Up to row 2^{19} :

$\{0, 1, 2, 4, 6, 8, 12, 16, 20, 24, 32, 40, 48, 56, 64, 72, 80, 96, \dots, 65536\}$

Conjecture

Fix n and odd r .

If $a_{16,r}(n)$ is divisible by 3, then it is also divisible by 2.

If $a_{16,r}(n)$ is divisible by 5, then it is also divisible by 2^2 .

If $a_{16,r}(n)$ is divisible by 7, then it is also divisible by 2^3 .

If $a_{16,r}(n)$ is divisible by 11, then it is also divisible by 2^5 .

If $a_{16,r}(n)$ is divisible by 13, then it is also divisible by 2^6 .

If $a_{16,r}(n)$ is divisible by 17, then it is also divisible by 2^4 .

If $a_{16,r}(n)$ is divisible by 31, then it is also divisible by 2^5 .

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Second question

The “high school dream”

$$(a + b)^n = a^n + b^n$$

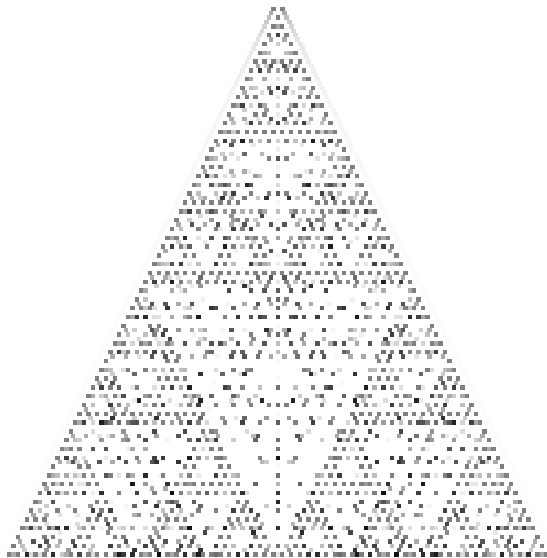
actually works sometimes.

Namely, $(a + b)^p \equiv a^p + b^p \pmod{p}$, since $\binom{p}{m} \equiv 0 \pmod{p}$ for $1 \leq m \leq p - 1$.

How badly does the high school dream fail for non-primes?

What is $\binom{n}{m} \pmod{n}$?

$$\binom{n}{m} \pmod n$$



Rather than fixing n and varying m , let us fix m and vary n .

Let $\text{frac}(x)$ be the fractional part of x .

Rather than looking at $\binom{n}{m} \pmod n$, let us look at $\text{frac}(\frac{1}{n} \binom{n}{m})$.

For fixed $m \geq 1$, $\text{frac}(\frac{1}{n} \binom{n}{m}) = \text{frac}(\frac{1}{m} \binom{n-1}{m-1})$ is periodic.

Let δ_S be 1 if S is true and 0 if S is false.

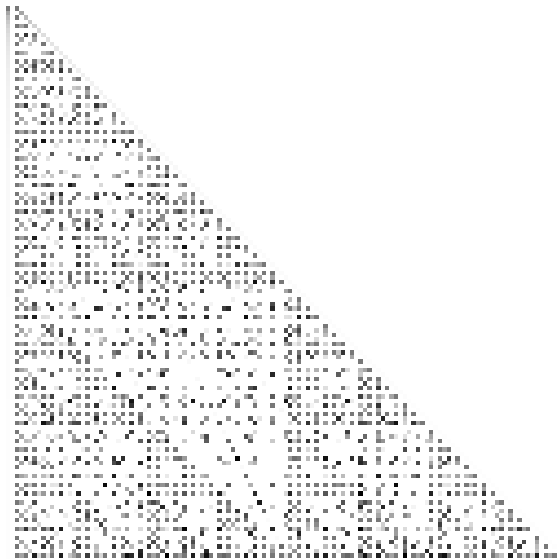
Proposition

Let p be a prime, and let $n \geq 1$. Then

$$\text{frac}\left(\frac{1}{n} \binom{n}{p}\right) = \frac{1}{p} \delta_{p|n} = \begin{cases} 0 & \text{if } p \nmid n \\ 1/p & \text{if } p \mid n. \end{cases}$$

So $\binom{n}{p} \equiv \frac{n}{p} \delta_{p|n} \pmod n$.

$$\binom{n}{m} \pmod n$$



$$\binom{n}{2p} \pmod n$$

What about non-primes? What is $\text{frac}(\frac{1}{n} \binom{n}{2p})$?

Conjecture

Let $p \neq 2$ be a prime, and let $n \geq 1$. Then

$$\text{frac}\left(\frac{1}{n} \binom{n}{2p}\right) = \text{frac}\left(\frac{(p+1)(n/p-1)}{2p}\right) \delta_{p|n} + \frac{1}{2} \delta_{\text{exceptional}_{2p}(n)}.$$

... where $m_l \cdots m_1 m_0$ is the binary representation of $m = 2p$ and $\text{exceptional}_{2p}(n)$ is the statement that

$$n \equiv 2p + \sum_{i=0}^l m_i 2^i \left\lfloor \frac{j}{2^{|\text{mod}(2p, 2^{i+1})|_0 - \delta_{i=1}}} \right\rfloor \pmod{2^{\lfloor \log_2(2p) \rfloor + 1}}$$

for some $0 \leq j \leq 2^{\lfloor 2p \rfloor_0} - 1$.

Crazy condition

The expression

$$\sum_{i=0}^l m_i 2^i \left\lfloor \frac{j}{2^{\lfloor \log_2(2p, 2^{i+1}) \rfloor_0 - \delta_{i=1}}} \right\rfloor$$

is *almost* the product of $2p$ and $j \dots$

For example, let $p = 197$ and $j = 27$.

$$\begin{array}{r} 110001010_2 \\ \times 11011_2 \\ \hline 11011 \\ 11011 \\ 11011 \\ + 11011 \\ \hline 10100110001110_2 \end{array}$$

Crazy condition

The expression

$$\sum_{i=0}^l m_i 2^i \left\lfloor \frac{j}{2^{\lfloor \log_2(2p, 2^{i+1}) \rfloor_0 - \delta_{i=1}}} \right\rfloor$$

is *almost* the product of $2p$ and $j \dots$

\dots except we delete some digits:

$$\begin{array}{r} 110001010_2 \\ * \quad 11011_2 \\ \hline 11011 \\ 1101\cancel{1} \\ \cancel{11011} \\ + \quad \cancel{11011} \\ \hline 1100110_2 \end{array}$$