Toward a language theoretic proof of the four color theorem

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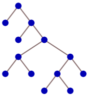
outline of the talk

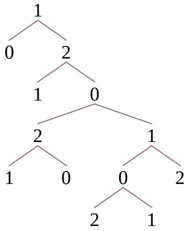
1. introduction
2. parameterized families of trees
3. reducing a pair of trees

the context-free grammar G

start symbols: $0, 1, 2$
formation rules: $0 \to 12$, $0 \to 21$, $1 \to 02$, $1 \to 20$, $2 \to 01$, $2 \to 10$

An n-leaf tree T **parses** a length-n word w on $\{0, 1, 2\}$
if T is a valid derivation tree for w under G.

For example, the tree  parses 0110212:



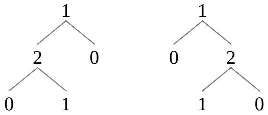
The set of possible derivation trees under G is the set of binary trees.

ambiguity

The grammar G is ambiguous;
there exist distinct trees that parse a common word.

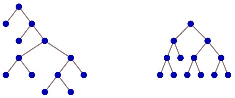
The trees

both parse 010:



another example

The trees



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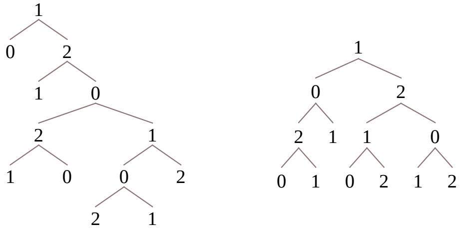
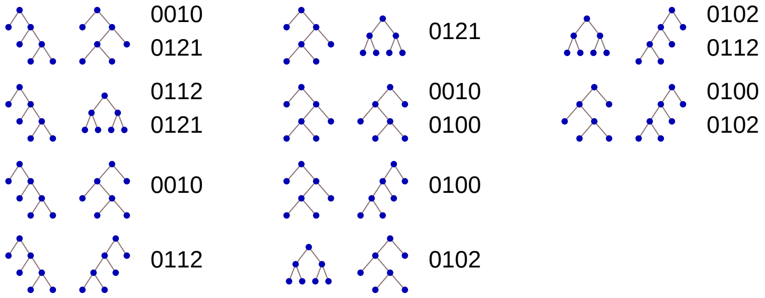


a much stronger statement

Theorem

Let $n \geq 1$, and let T_1 and T_2 be n-leaf binary trees. Then T_1 and T_2 parse a common word under G.



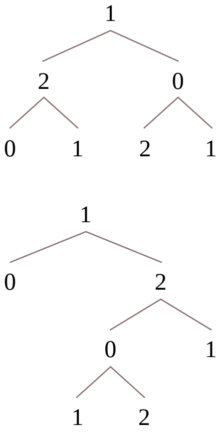
Theorem (Louis Kauffman, 1990)

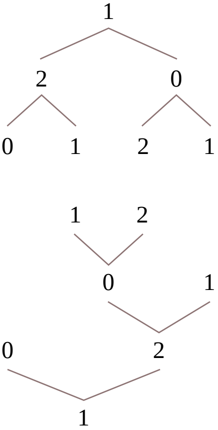
The following are equivalent.

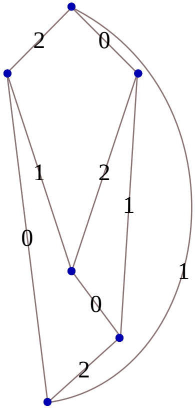
- *Every pair of n-leaf binary trees parses a common word under G.*
- *Every planar map is four-colorable.*

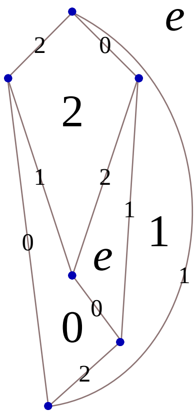
Perhaps an enumerative or language theoretic approach will lead to a proof of the four color theorem that is shorter than known proofs.

sketch of correspondence



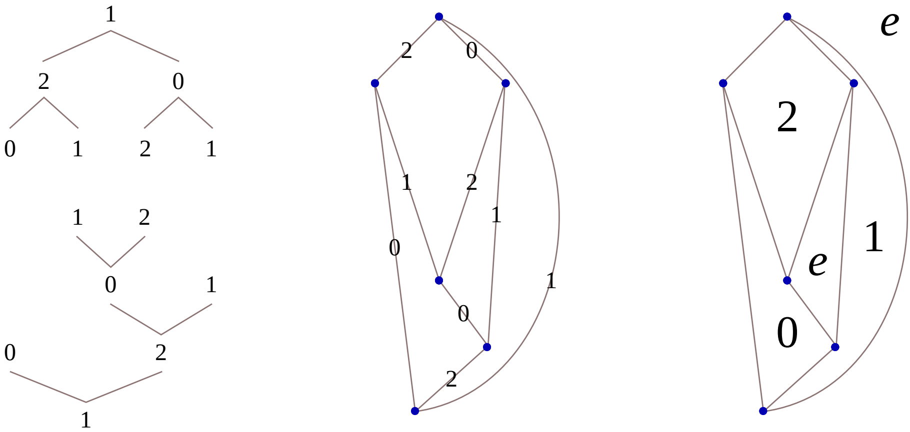






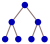
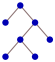
$\{e, 0, 1, 2\}$ is the Klein 4-group.

sketch of correspondence



equivalence classes of parse words

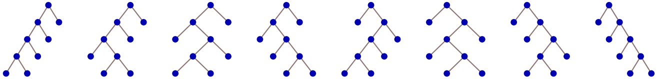
Let ParseWords(T_1, T_2) be the set of equivalence classes of words parsed by both trees T_1 and T_2.

For example, ParseWords(, ) $= \{0121\}$.

The four color theorem is equivalent to the statement that for every pair of n-leaf binary trees T_1 and T_2 we have ParseWords(T_1, T_2) $\neq \{\}$.

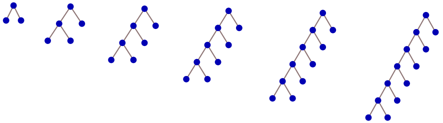
outline of the talk

A path tree is a binary tree with at most two vertices on each level.



lll llr lrl lrr rll rlr rrl rrr

Let LeftCombTree(n) be the n-leaf path tree corresponding to l^{n-2}.



$\ldots$

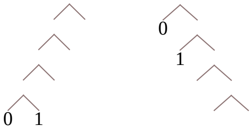
Let RightCombTree(n) be the n-leaf path tree corresponding to r^{n-2}.

Theorem

$$\mathrm{ParseWords}(\mathrm{LeftCombTree}(n), \mathrm{RightCombTree}(n)) =$$

$$\begin{cases} \{01^{n-2}2\} & \textit{if } n \geq 2 \textit{ is even} \\ \{01^{n-2}0\} & \textit{if } n \geq 3 \textit{ is odd.} \end{cases}$$

Proof by example.



a pair of comb trees

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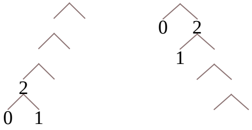

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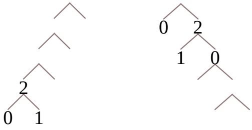

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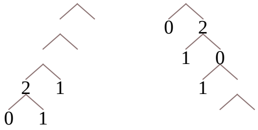

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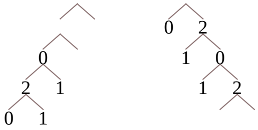

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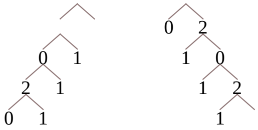

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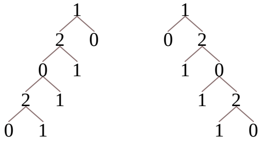
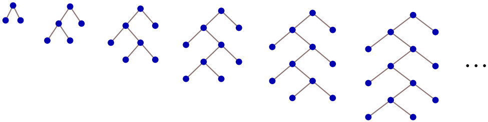
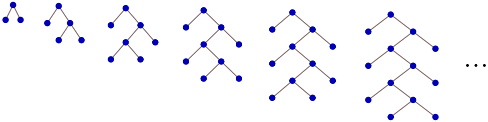


crooked trees

Let LeftCrookedTree(n) be the path tree corresponding to $(lr)^{(n-2)/2}$.



Let RightCrookedTree(n) be the path tree corresponding to $(rl)^{(n-2)/2}$.

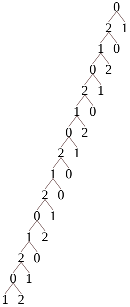


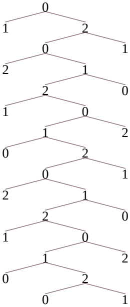
a comb tree and a crooked tree

Theorem

ParseWords(LeftCombTree(n), RightCrookedTree(n)) $=$

$$\begin{cases} \left\{ \operatorname{mod}(1-n,3)\ \left((012)^{n/6}\right)^R (012)^{(n-2)/6} \right\} & \textit{if } n \geq 2 \textit{ is even} \\ \left\{ \operatorname{mod}(1-n,3)\ \left((012)^{(n-3)/6}\right)^R (012)^{(n+1)/6} \right\} & \textit{if } n \geq 3 \textit{ is odd.} \end{cases}$$





a pair of crooked trees

The number of parse words is generally not constant.

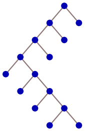
Theorem

For $n \geq 2$,

$$|\text{ParseWords}(\text{LeftCrookedTree}(n), \text{RightCrookedTree}(n))| = 2^{\lfloor n/2 \rfloor - 1}.$$

a two-parameter family

Let LeftCombTree(m, n) and RightCombTree(m, n) be the $(m+n)$-leaf path trees corresponding to $l^m r^{n-2}$ and $r^m l^{n-2}$.

For example, LeftCombTree$(3, 5) =$  .

Theorem

For $m \geq 1$, $k \geq 1$, and $n \geq k+2$,

$$b(m,k) = |\text{ParseWords}(\text{LeftCombTree}(m,n), \text{RightCombTree}(k, m+n-k))|$$

is independent of n. Moreover,

$$(M-2)(M-1)(M+1)\, b(m,k) = 0.$$

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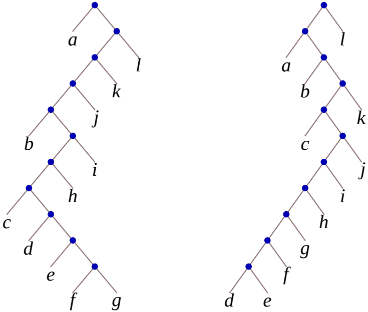
decomposable pairs

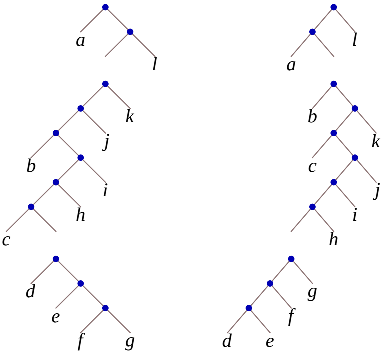
Consider the pair



If two trees have subtrees with the same sets of leaves, we can decompose the pair into smaller pairs.

decomposable pairs

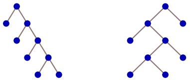
Breaking the trees as



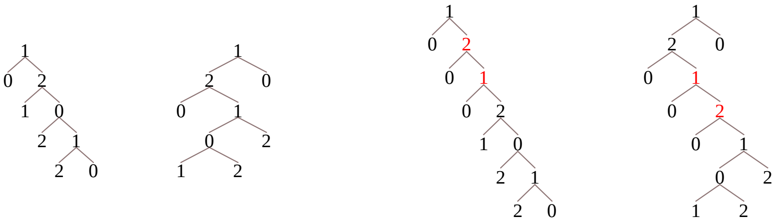
produces the same partition $\{\{a, l\}, \{b, c, h, i, j, k\}, \{d, e, f, g\}\}$ of the leaves in both trees.

extending a parse word . . .

Consider the pair



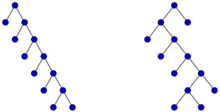
which parses 01220. “Triplicate” the first leaf:

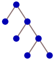


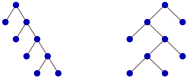
We have extended the parse word for the smaller pair to a parse word for a larger pair.

. . . reducing a parse word

Since the (larger) pair



contains the right comb  in leaves 1–3, it is reducible to



Theorem

If a pair of n-leaf (not necessarily path) trees has three consecutive leaves that appear in a comb structure in both trees, then the pair is reducible.

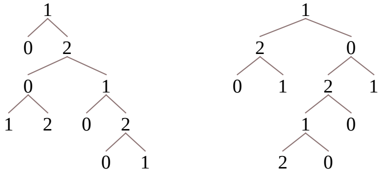
In particular, the three leaves receive the same label for some parse word.

"mutual crookedness"

However, something stronger appears to be true.

Conjecture

If a pair of n-leaf trees has two consecutive leaves that appear in a comb structure in both trees, then there is a parse word in which the two leaves receive the same label.



But there is no obvious relationship between the parse word of the original pair and the parse word of the "reduced" pair!

concluding thoughts

- To prove the “four color theorem for path trees” it suffices to consider indecomposable, “weakly mutually crooked” pairs of path trees.
- Existing proofs of the four color theorem successfully use the notion of reducibility.
  Should the language theoretic approach also pursue it?
- Since the number of parse words of LeftCombTree(m, n) and RightCombTree$(k, m+n-k)$ satisfies such a simple recurrence, looking for generalizations seems like a promising direction.