The structure of equivalence classes in F_2

Eric Rowland, joint work with Bobbe Cooper

Laboratoire de Combinatoire et d'Informatique Mathématique
Université du Québec à Montréal

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Outline

Automorphisms

- $L_2 = \{a, b, \overline{a}, \overline{b}\}$, where $\overline{a} = a^{-1}$ and $\overline{b} = b^{-1}$.
- The free group on two generators:
  $F_2 = \langle a, b \rangle = \{w_1 \cdots w_\ell \in L_2^* : w_i \neq w_{i+1}^{-1} \text{ for } 1 \leq i \leq \ell - 1\}$
  $= \{\epsilon, a, b, \overline{a}, \overline{b}, aa, ab, a\overline{b}, ba, bb, b\overline{a}, \overline{a}b, \overline{a}\overline{a}, \overline{a}\overline{b}, \overline{b}a, \overline{b}\overline{a}, \overline{b}\overline{b}, \dots\}$.

Example

Let $\pi(a) = b, \pi(b) = \overline{a}$.
π extends to an automorphism of F_2: $\pi(\overline{a}) = \overline{b}$ and $\pi(\overline{b}) = a$.
For example, $\pi(aaab\overline{a}b) = bbb\overline{a}\overline{b}\overline{a}$.

A permutation is an automorphism that permutes L_2.

Example

Let ϕ be the inner automorphism $\phi(w) = \overline{a}bw\overline{b}a$.
Then $\phi(aaab\overline{a}b) = \overline{a}baaab$ and $\phi(a) = \overline{a}bab\overline{b}a$.

Automorphisms

Equivalence under

- permutations ($aaab\bar{a}b \sim bbb\bar{a}\bar{b}\bar{a} \sim bbba\bar{b}a \sim \cdots$)

and

- inner automorphisms ($aaab\bar{a}b \sim aab\bar{a}ba \sim ab\bar{a}baa \sim \cdots$)

is cosmetic.

It is natural to work modulo Inn F_2 and consider cyclic words.

We'll also look at equivalence classes of words modulo permutations.

One-letter automorphisms

Definition

A one-letter automorphism of F_2 is an automorphism which maps $x \mapsto x$ and $y \mapsto yx$ for some $x, y \in L_2$ with $y \notin \{x, \overline{x}\}$.

We will denote by ϕ_{yx} the one-letter automorphism $x \mapsto x$, $y \mapsto yx$. The inverse automorphism maps $x \mapsto x$ and $y \mapsto y\overline{x}$.

Example

Let $\phi = \phi_{ba}$. Then $\phi(b) = ba, \phi(\overline{b}) = \overline{a}\overline{b}$.
$\phi(a\overline{b}) = a\overline{a}\overline{b} = \overline{b}$, so $a\overline{b}$ is not minimal.

Definition

A word $w \in F_2$ is minimal if $|w| \leq |\phi(w)|$ for all $\phi \in \operatorname{Aut} F_2$.

Whitehead's theorem

We write $w \sim v$ if $\phi(w) = v$ for some automorphism ϕ.

Theorem (Whitehead, 1936)

If $w, v \in F_2$ such that $w \sim v$ and v is minimal, then there exists a sequence $\phi_1, \phi_2, \ldots, \phi_m$ of one-letter automorphisms and a permutation π such that

- $\pi\phi_m \cdots \phi_2\phi_1(w) = v$ *as cyclic words and*
- $|\phi_{k+1}\phi_k \cdots \phi_2\phi_1(w)| < |\phi_k \cdots \phi_2\phi_1(w)|$ *for* $0 \leq k \leq m-1$.

The set of one-letter automorphisms is finite. (There are 8.)

Corollary

Determining minimality of a word in F_2 is decidable.
Determining equivalence of two words in F_2 is decidable.

Computing equivalence classes

Corollary

Given n, there is an algorithm for computing representatives of all equivalence classes of F_2 under $\sim$ containing a word of length $\leq n$.

We computed all classes containing words of length ≤ 20,
recording minimal words up to permutations and inner automorphisms.
(There are 14 027 794 classes whose minimal words have length 20.)

Example

$$\begin{aligned} W = \{ & aa, bb, \overline{aa}, \overline{bb}, abab, abb\bar{a}, a\bar{b}a\bar{b}, a\bar{b}\bar{b}\bar{a}, baa\bar{b}, baba, b\bar{a}b\bar{a}, b\bar{a}\bar{a}\bar{b}, \\ & \bar{a}bba, \bar{a}b\bar{a}b, \bar{a}\bar{b}\bar{a}b, \bar{a}\bar{b}\bar{b}a, \bar{b}aab, \bar{b}a\bar{b}a, \bar{b}\bar{a}\bar{a}b, \bar{b}\bar{a}\bar{b}\bar{a}, aabaab, \ldots \}. \end{aligned}$$

The minimal words in W are $\{aa, bb, \overline{aa}, \overline{bb}\}$, and the lex least is aa.

Equivalence classes

0.1	ϵ
1.1	a
2.1	aa
3.1	aaa
4.1	$aaaa$
4.2	$ab\bar{a}\bar{b}$
4.3	$aabb$
	$aba\bar{b}$
5.1	$aaaaa$
5.2	$aaba\bar{b}$
5.3	$aab\bar{a}\bar{b}$
5.4	$aaabb$
	$aab\bar{a}b$

6.1	$aaaaaa$
6.2	$aaaba\bar{b}$
6.3	$aaabbb$
6.4	$aaab\bar{a}\bar{b}$
6.5	$aabaa\bar{b}$
6.6	$aaba\bar{b}\bar{b}$
6.7	$aabba\bar{b}$
6.8	$aabb\bar{a}\bar{b}$
6.9	$aab\bar{a}\bar{a}\bar{b}$
6.10	$aaaabb$
	$aaab\bar{a}b$
	$aab\bar{a}\bar{a}b$
7.1	$aaaaaaa$
7.2	$aaaaba\bar{b}$
7.3	$aaaabbb$

7.4	$aaaab\bar{a}\bar{b}$
7.5	$aaabaa\bar{b}$
7.6	$aaaba\bar{b}\bar{b}$
7.7	$aaabba\bar{b}$
7.8	$aaabb\bar{a}b$
7.9	$aaabb\bar{a}\bar{b}$
7.10	$aaab\bar{a}bb$
7.11	$aaab\bar{a}\bar{a}\bar{b}$
7.12	$aaab\bar{a}\bar{b}\bar{b}$
7.13	$aabaa\bar{b}\bar{b}$
7.14	$aabba\bar{b}\bar{b}$
7.15	$aabb\bar{a}\bar{a}\bar{b}$
7.16	$aaaaabb$
	$aaaab\bar{a}b$
	$aaab\bar{a}\bar{a}b$

Equivalence class size

- Myasnikov and Shpilrain (2003): The number of minimal words equivalent to minimal $w \in F_2$ is bounded by a polynomial in $|w|$.
- Lee (2006): The same is true for minimal $w \in F_n$ under a local condition on w.

These results imply upper bounds on the time required to determine whether $w, v \in F_n$ are equivalent.

Outline

Subword counts

Definition

Let $(v)_w$ denote the number of (possibly overlapping) occurrences of v and v^{-1} in the cyclic word w.

Example

Let $w = aa\overline{b}\overline{b}\overline{a}bab\overline{b}a$. The length-2 subword counts are $(aa)_w = 2$, $(bb)_w = 1$, $(ab)_w = 1 = (ba)_w$, and $(a\overline{b})_w = 2 = (\overline{b}a)_w$.

Lemma

If $x, y \in L_2$, then $(xy)_w = (yx)_w$.

Characterization of minimal words

Recall that the one-letter automorphism ϕ_{yx} maps $x \mapsto x$ and $y \mapsto yx$.

ϕ_{yx} causes cancellations in the subword $y\overline{x}$ and its inverse.
ϕ_{yx} causes additions in the subwords yx and yy and their inverses.

Lemma

w is minimal if and only if, for each pair of letters x and y with $x \notin \{y, \overline{y}\}$, we have $(y\overline{x})_w \leq (yx)_w + (yy)_w$.

Theorem

w is minimal if and only if $|(ab)_w - (a\overline{b})_w| \leq \min((aa)_w, (bb)_w)$.

Equivalence classes

0.1	ϵ
1.1	a
2.1	aa
3.1	aaa
4.1	$aaaa$
4.2	$ab\bar{a}\bar{b}$
4.3	$aabb$
	$aba\bar{b}$
5.1	$aaaaa$
5.2	$aaba\bar{b}$
5.3	$aab\bar{a}\bar{b}$
5.4	$aaabb$
	$aab\bar{a}b$

6.1	$aaaaaa$
6.2	$aaaba\bar{b}$
6.3	$aaabbb$
6.4	$aaab\bar{a}\bar{b}$
6.5	$aabaa\bar{b}$
6.6	$aaba\bar{b}\bar{b}$
6.7	$aabba\bar{b}$
6.8	$aabb\bar{a}\bar{b}$
6.9	$aab\bar{a}\bar{a}\bar{b}$
6.10	$aaaabb$
	$aaab\bar{a}b$
	$aab\bar{a}\bar{a}b$
7.1	$aaaaaaa$
7.2	$aaaaba\bar{b}$
7.3	$aaaabbb$

7.4	$aaaab\bar{a}\bar{b}$
7.5	$aaabaa\bar{b}$
7.6	$aaaba\bar{b}\bar{b}$
7.7	$aaabba\bar{b}$
7.8	$aaabb\bar{a}b$
7.9	$aaabb\bar{a}\bar{b}$
7.10	$aaab\bar{a}bb$
7.11	$aaab\bar{a}\bar{a}\bar{b}$
7.12	$aaab\bar{a}\bar{b}\bar{b}$
7.13	$aabaa\bar{b}\bar{b}$
7.14	$aabba\bar{b}\bar{b}$
7.15	$aabb\bar{a}\bar{a}\bar{b}$
7.16	$aaaaabb$
	$aaaab\bar{a}b$
	$aaab\bar{a}\bar{a}b$

Equivalence classes

0.1	ϵ
1.1	a
2.1	aa
3.1	aaa
4.1	$aaaa$
4.2	$ab\bar{a}\bar{b}$
4.3	$aabb$
	$aba\bar{b}$
5.1	$aaaaa$
5.2	$aaba\bar{b}$
5.3	$aab\bar{a}\bar{b}$
5.4	$aaabb$
	$aab\bar{a}b$

6.1	$aaaaaa$
6.2	$aaaba\bar{b}$
6.3	$aaabbb$
6.4	$aaab\bar{a}\bar{b}$
6.5	$aabaa\bar{b}$
6.6	$aaba\overline{bb}$
6.7	$aabba\bar{b}$
6.8	$aabb\bar{a}\bar{b}$
6.9	$aab\overline{aa}\bar{b}$
6.10	$aaaabb$
	$aaab\bar{a}b$
	$aab\overline{aa}b$
7.1	$aaaaaaa$
7.2	$aaaaba\bar{b}$
7.3	$aaaabbb$

7.4	$aaaab\bar{a}\bar{b}$
7.5	$aaabaa\bar{b}$
7.6	$aaaba\overline{bb}$
7.7	$aaabba\bar{b}$
7.8	$aaabb\bar{a}b$
7.9	$aaabb\bar{a}\bar{b}$
7.10	$aaab\bar{a}bb$
7.11	$aaab\overline{aa}\bar{b}$
7.12	$aaab\bar{a}\overline{bb}$
7.13	$aabaa\overline{bb}$
7.14	$aabba\overline{bb}$
7.15	$aabb\overline{aa}\bar{b}$
7.16	$aaaaabb$
	$aaaab\bar{a}b$
	$aaab\overline{aa}b$

Growing words

Definition

A child of $w \neq \epsilon$ is a word obtained by duplicating a letter in w. Also define the words $a, \overline{a}, b, \overline{b}$ to be children of ϵ.

Example

The children of $aaab\overline{a}b$ are $aaaab\overline{a}b$, $aaabb\overline{a}b$, $aaab\overline{a}\overline{a}b$, and $aaab\overline{a}bb$.

A child of a minimal word w is necessarily minimal, since

$$|(ab)_w - (a\overline{b})_w| \leq \min((aa)_w, (bb)_w).$$

Root words

Definition

A root word is a minimal word that is not a child of any minimal word.

Root words are new minimal words with respect to duplicating a letter.

Example

The minimal word *aabb* is a root word, since neither of its parents *abb* and *aab* is minimal.

Example

The minimal words $ab a\bar{b}$ and $ab\bar{a}\bar{b}$ are root words. They are not children of any minimal word; in particular they have no subword *xx*.

A word w is alternating if $(aa)_w = (bb)_w = 0$.
More generally, any alternating minimal word is a root word.

Characterization of root words

Root words refine the notion of minimal words.

Theorem

w is a root word if and only if $|(ab)_w - (a\bar{b})_w| = (aa)_w = (bb)_w$.

Proof.

Recall: w is minimal if and only if $|(ab)_w - (a\bar{b})_w| \leq \min((aa)_w, (bb)_w)$.
A minimal word w is a root word if and only if replacing any xx by x in w produces a non-minimal word. □

Example

$w = (ab\bar{a}\bar{b})^n$ is a root word: $|(ab)_w - (a\bar{b})_w| = 0 = (aa)_w = (bb)_w$.

Properties of root words

Theorem

If w is a root word, then $|w|$ is divisible by 4.

The property of being a root word is preserved under automorphisms.

Theorem

If w is a root word, $w \sim v$, and $|w| = |v|$, then v is a root word.

A class W whose minimal words are root words is a root class.

Root classes

0.1	ϵ
1.1	a
2.1	aa
3.1	aaa
4.1	$aaaa$
4.2	$ab\bar{a}\bar{b}$
4.3	$aabb$ $aba\bar{b}$
5.1	$aaaaa$
5.2	$aaba\bar{b}$
5.3	$aab\bar{a}\bar{b}$
5.4	$aaabb$ $aab\bar{a}b$

6.1	$aaaaaa$
6.2	$aaaba\bar{b}$
6.3	$aaabbb$
6.4	$aaab\bar{a}\bar{b}$
6.5	$aabaa\bar{b}$
6.6	$aaba\bar{b}\bar{b}$
6.7	$aabba\bar{b}$
6.8	$aabb\bar{a}\bar{b}$
6.9	$aab\bar{a}\bar{a}\bar{b}$
6.10	$aaaabb$ $aaab\bar{a}b$ $aab\bar{a}\bar{a}b$
7.1	$aaaaaaa$
7.2	$aaaaba\bar{b}$
7.3	$aaaabbb$

7.4	$aaaab\bar{a}\bar{b}$
7.5	$aaabaa\bar{b}$
7.6	$aaaba\bar{b}\bar{b}$
7.7	$aaabba\bar{b}$
7.8	$aaabb\bar{a}b$
7.9	$aaabb\bar{a}\bar{b}$
7.10	$aaab\bar{a}bb$
7.11	$aaab\bar{a}\bar{a}\bar{b}$
7.12	$aaab\bar{a}\bar{b}\bar{b}$
7.13	$aabaa\bar{b}\bar{b}$
7.14	$aabba\bar{b}\bar{b}$
7.15	$aabb\bar{a}\bar{a}\bar{b}$
7.16	$aaaaabb$ $aaaab\bar{a}b$ $aaab\bar{a}\bar{a}b$

Outline

Equivalence class graph

Let $\Gamma(W)$ be a directed graph whose vertices are the minimal words of W up to permutations and inner automorphisms.

We draw an edge $w \to v$ for each one-letter automorphism ϕ such that $\phi(w)$ and v differ only by a permutation and an inner automorphism.

There are 8 one-letter automorphisms, but only 4 classes modulo $\operatorname{Inn} F_2$, since $\phi_{yx} = (x \mapsto yx\overline{y}) \circ \phi_{\overline{yx}}$.

Example

Consider equivalence class 4.3 containing $aabb$ and $aba\bar{b}$.
Apply the 4 one-letter automorphisms to $aabb$:

$$\begin{array}{llcll} a \mapsto ab : & ababbb & \qquad & b \mapsto ba : & aababa \\ a \mapsto a\bar{b} : & a\bar{b}ab & & b \mapsto b\bar{a} : & ab\bar{a}b \end{array}$$

Apply the 4 one-letter automorphisms to $aba\bar{b}$:

$$\begin{array}{llcll} a \mapsto ab : & abba & \qquad & b \mapsto ba : & aba\bar{b} \\ a \mapsto a\bar{b} : & aa\bar{b}\bar{b} & & b \mapsto b\bar{a} : & aba\bar{b} \end{array}$$

Therefore $\Gamma(W)$ is

$$\begin{tikzcd} aabb \arrow[r, shift left=3] \arrow[r, shift left=1] & aba\bar{b} \arrow[l, shift left=1] \arrow[l, shift left=3] \arrow[loop above] \arrow[loop below] \end{tikzcd}.$$

Whitehead's theorem implies that $\Gamma(W)$ is connected.

Khan's thesis

Khan (2004) studied the structure of $\Gamma(W)$.

Showing that a large number of subgraphs are forbidden puts bounds on the size of a spanning tree of $\Gamma(W)$.

Theorem (Khan)

If W is an equivalence class whose minimal words have length $n \geq 10$ and $|V(\Gamma(W))| \geq 4373$, then $|V(\Gamma(W))| \leq n - 5$.

Equivalence of $w, v \in F_2$ can be determined in time $O(\max(|w|, |v|)^2)$.

Direct approach

Our work recasts Khan's results, with shorter proofs and sharper constants:

If W is an equivalence class whose minimal words have length $n \geq 9$, then $|V(\Gamma(W))| \leq n - 5$.

Khan identified two types of legal spanning trees:

(i) trees with at most $n - 5$ vertices and simple edge structure
(ii) trees with size bounded by some constant

These correspond to non-root classes and root classes.

Structure of non-root classes

Theorem

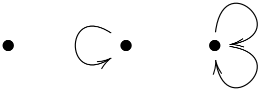
Let W be a non-root class. Then $\Gamma(W)$ has one of the following forms.

(P1)

(P2)

(P3)

(P1)–(P3) include degenerate forms:



Simple path graph

Let $\phi = \phi_{b\bar{a}}$, which maps $b \mapsto b\bar{a}$.

Example

$$aaabaabbb \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} aabab\bar{a}b\bar{a}b \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} abb\bar{a}\bar{a}b\bar{a}\bar{a}b$$

ϕ shrinks subwords $ba^m b$ and extends subwords $b\bar{a}^m b$.
…and leaves subwords $ba^{\pm m}\bar{b}$ fixed.

Example

$$aaab\bar{a}babb \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} aab\bar{a}bab\bar{a}b \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} ab\bar{a}bab\bar{a}\bar{a}b \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} b\bar{a}bab\bar{a}\bar{a}\bar{a}b$$

Vertices with outdegree 2 have $(bb)_w = 0$.

Folded path graph

Example

Again let $\phi = \phi_{b\bar{a}}$.

$$aaaabb \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} aaab\bar{a}b \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} aab\bar{a}\bar{a}b \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} ab\bar{a}\bar{a}\bar{a}b \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} b\bar{a}\bar{a}\bar{a}\bar{a}b$$

But $\phi(aab\bar{a}\bar{a}b) = ab\bar{a}\bar{a}\bar{a}b = \pi(aaab\bar{a}b)$, where $\pi(a) = \bar{a}$ and $\pi(b) = b$.

$$\begin{tikzcd} aaaabb \arrow[r, shift left=2, "\phi"] & aaab\bar{a}b \arrow[l, shift left=2, "\phi^{-1}"] \arrow[r, shift left=3, "\phi"] & aab\bar{a}\bar{a}b \arrow[l, "\pi\phi" description] \arrow[l, shift left=3, "\phi^{-1}"] \end{tikzcd} .$$

Broken symmetry occurs; even though $\phi^{-1}\pi(aaab\bar{a}b) = aab\bar{a}\bar{a}b$, $\phi^{-1}\pi$ does not contribute an edge to $\Gamma(W)$ since π applies before ϕ^{-1}.

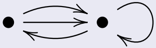
Structure of root classes

Theorem

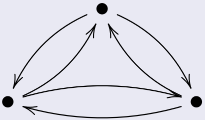
Let W be a root class with no alternating minimal word. Then $\Gamma(W)$ is one of the following graphs.

(R1)

(R2)



(R3)



Triangle graph

Example

$\phi_{a\bar{b}}(aaababbb) = a\bar{b}a\bar{b}aabb$.
$\phi_{b\bar{a}}(aaababbb) = aabb\bar{a}b\bar{a}b$.

$$\begin{tikzcd}
& a\bar{b}a\bar{b}aabb \arrow[dl, bend right, "\phi_{ab}"'] \arrow[dr, bend left, "\pi_2\phi_{ba}"] & \\
aaababbb \arrow[ur, "\phi_{a\bar{b}}"] \arrow[rr, bend left, "\phi_{b\bar{a}}"] & & aabb\bar{a}b\bar{a}b \arrow[ul, "\pi_1\phi_{ab}"'] \arrow[ll, bend left, "\phi_{ba}"]
\end{tikzcd}$$

$\phi_{ab}(aabb\bar{a}b\bar{a}b) = ababb\bar{a}\bar{a}b$.

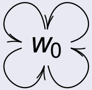
$\phi_{ba}(a\bar{b}a\bar{b}aabb) = \bar{b}\bar{b}aababa$.

Structure of root classes

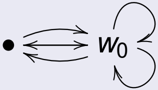
Theorem

Let W be a root class containing an alternating minimal word w_0. Then $\Gamma(W)$ is one of the following graphs.

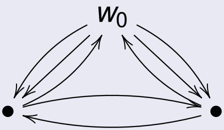
(R4)



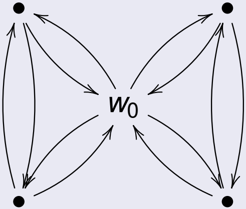
(R5)



(R6)



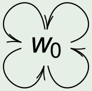
(R7)



Clover graph

Example

Let $w_0 = (ab\bar{a}\bar{b})^n$.
The automorphism $a \mapsto ab$ maps w_0 to $((ab)b(\bar{b}\bar{a})\bar{b})^n = (ab\bar{a}\bar{b})^n = w_0$.
Similarly, the other three one-letter automorphisms leave w_0 fixed:



In fact these are the only equivalence classes of type (R4):

If w_0 is an alternating minimal word whose graph has only 1 vertex, then for each automorphism $\phi : y \mapsto yx$ the word $\phi(w_0)$ is alternating.

Therefore $0 = (yy)_{\phi(w_0)} = (y\bar{x}y)_{w_0}$.

Summary

Non-root classes have simple path structure.

Root word classes have one of seven possible graphs.

Corollary

If W is a root class, then $\Gamma(W)$ has $1, 2, 3,$ or 5 vertices.

Moreover, each of the forms (P1)–(P3) and (R1)–(R7) occurs.

n	(P1)	(P2)	(P3)	(R1)	(R2)	(R3)	(R4)	(R5)	(R6)	(R7)
0	0	0	0	0	0	0	1	0	0	0
1	0	0	1	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0
4	0	0	1	0	0	0	1	1	0	0
5	0	1	3	0	0	0	0	0	0	0
6	4	0	6	0	0	0	0	0	0	0
7	10	1	5	0	0	0	0	0	0	0
8	22	0	8	1	2	3	1	3	1	2
9	81	5	15	0	0	0	0	0	0	0
10	298	4	38	0	0	0	0	0	0	0
11	855	7	49	0	0	0	0	0	0	0
12	2140	4	96	4	12	244	1	7	5	31
13	7040	29	155	0	0	0	0	0	0	0
14	22244	30	342	0	0	0	0	0	0	0
15	64774	49	553	0	0	0	0	0	0	0
16	175209	46	1104	11	70	10899	1	19	15	380
17	543631	185	1927	0	0	0	0	0	0	0
18	1649842	232	3892	0	0	0	0	0	0	0
19	4824825	343	6889	0	0	0	0	0	0	0
20	13535352	406	13592	35	400	473355	1	55	51	4547

The primary lemma

ϕ is level on w if $|\phi(w)| = |w|$.

Lemma

Suppose w is a minimal word such that ϕ_{yx} is level on w. Then

- *$\phi_{y\overline{x}}$ is level on w if and only if $(yy)_w = 0$,*
- *ϕ_{xy} is level on w if and only if w is a root word, and*
- *$\phi_{x\overline{y}}$ is level on w if and only if w is an alternating root word.*

Knowing whether w is alternating is significant information.

Future work

- Understand descendancy on the level of equivalence classes.
- Can equivalence classes be enumerated?
- For $10 \leq n \leq 20$,

  $$\frac{\#\text{ equivalence classes whose minimal words have length } n}{3^n} \approx .004.$$

  Does the limit exist?
- How much carries over to F_n?