The lexicographically least $5/4$-power-free word on $\mathbb{Z}_{\geq 0}$

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Joint Mathematics Meetings
AMS Special Session on Sequences, Words, and Automata
Denver, 2020–01–15

Avoiding squares


Axel Thue (1863–1922)

A square is a nonempty word of the form ww.
Are there arbitrarily long square-free words on $\{0, 1\}$?

Choose an order on $\{0, 1\}$ and try to construct one:

010☒

Infinite alphabet

What is the lexicographically least word on $\mathbb{Z}_{\geq 0}$ avoiding a pattern?

Theorem (Guay-Paquet–Shallit 2009)

Let $\varphi(n) = 0\,(n+1)$.
The lexicographically least square-free word on $\mathbb{Z}_{\geq 0}$ is $\varphi^\infty(0)$.

$$\begin{aligned}
\varphi(0) &= 01 \\
\varphi^2(0) &= 0102 \\
\varphi^3(0) &= 01020103 \\
&\vdots \\
\varphi^\infty(0) &= 0102010301020104010201030102010 5\cdots
\end{aligned}$$

More generally, let $a \geq 2$. Let $\varphi(n) = 0^{a-1}(n+1)$.
The lexicographically least a-power-free word on $\mathbb{Z}_{\geq 0}$ is $\varphi^\infty(0)$.

Fractional powers

$(0111)^{3/2} = 011101$ is a $\frac{3}{2}$-power.

Definition

If $v = v_0 v_1 \cdots v_{\ell-1}$ is a nonempty word whose length ℓ is divisible by b, the $\frac{a}{b}$-power of v is

$$v^{a/b} := v^{\lfloor a/b \rfloor} v_0 v_1 \cdots v_{\ell \cdot (a/b - \lfloor a/b \rfloor) - 1}.$$

$\frac{5}{4}$-powers look like $xyx = (xy)^{5/4}$ where $|y| = 3|x|$.

Notation

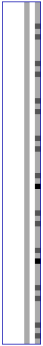
For $\frac{a}{b} > 1$, let $\mathbf{w}_{a/b}$ be the lex. least $\frac{a}{b}$-power-free word on $\mathbb{Z}_{\geq 0}$.

We assume $\gcd(a, b) = 1$ and $1 < \frac{a}{b} < 2$.

Avoiding 5/3-powers

$$\mathbf{w}_{5/3} = 000010100001010000101000010100001020000101\cdots$$

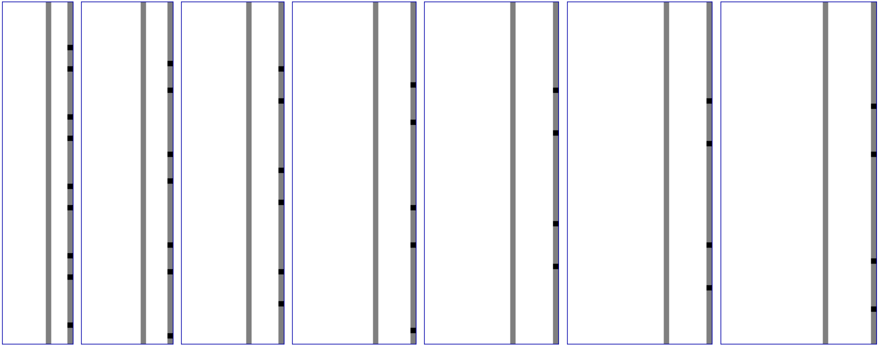
$$\begin{aligned}\mathbf{w}_{5/3} = {} & 0000101 \\ & 0000101 \\ & 0000101 \\ & 0000101 \\ & 0000102 \\ & 0000101 \\ & 0000102 \\ & 0000101 \\ & \vdots\end{aligned}$$



$$w(7i+6) = w(i) + 1$$

$\mathbf{w}_{5/3} = \varphi^\infty(0)$, where $\varphi(n) = 000010(n+1)$ is a 7-uniform morphism.

A family related to $\mathbf{w}_{5/3}$



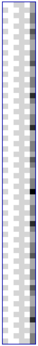
$\mathbf{w}_{9/5}$ $\mathbf{w}_{12/7}$ $\mathbf{w}_{13/7}$ $\mathbf{w}_{16/9}$ $\mathbf{w}_{17/9}$ $\mathbf{w}_{19/11}$ $\mathbf{w}_{20/11}$

Theorem (Pudwell–Rowland 2018)

Let $\frac{5}{3} \leq \frac{a}{b} < 2$ with b odd. Then $\mathbf{w}_{a/b} = \varphi^{\infty}(0)$, where $\varphi(n) = 0^{a-1}\,1\,0^{a-b-1}\,(n+1)$ is a $(2a-b)$-uniform morphism.

Avoiding $3/2$-powers

$$\mathbf{w}_{3/2} = 001102100112001103100113001102100114001103\cdots$$



Use two kinds of letters.
Alphabet: $\Sigma_2 = \{n_j : n \in \mathbb{Z}, j \in \{0,1\}\}$
Coding: $\tau(n_j) = n$

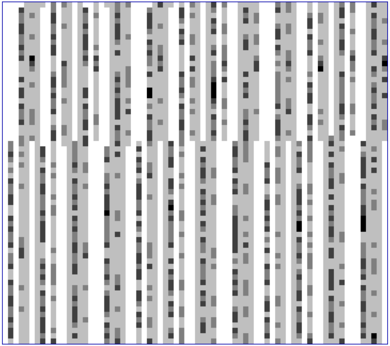
6-uniform morphism:
$\varphi(n_0) = 0_0 0_1 1_0 1_1 0_0 (n+2)_1$
$\varphi(n_1) = 1_0 0_1 0_0 1_1 1_0 (n+2)_1$

Theorem (Rowland–Shallit 2012)

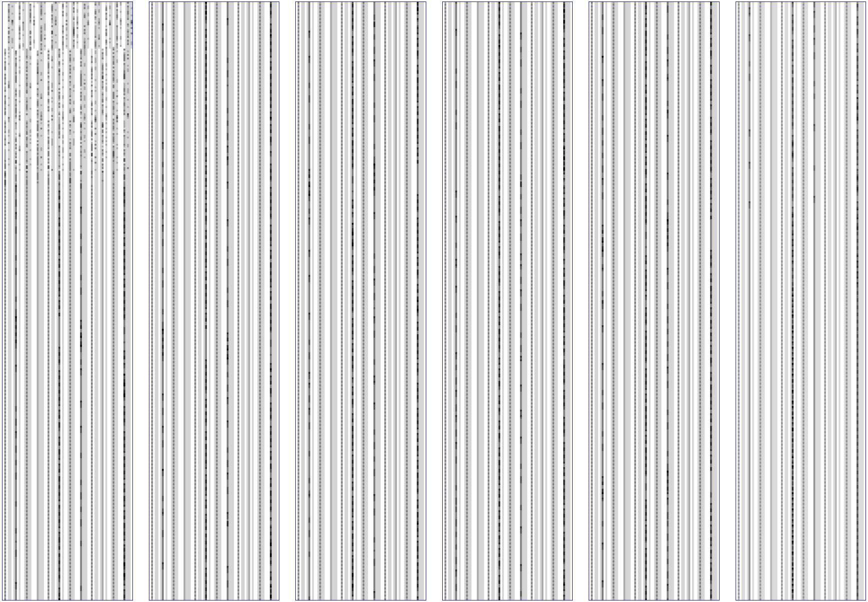
$\mathbf{w}_{3/2} = \tau(\varphi^\infty(0_0))$.

$\mathbf{w}_{5/4}$ wrapped into 72 columns

$$\mathbf{w}_{5/4} = 000011110202101001011212000013110102101302\cdots$$



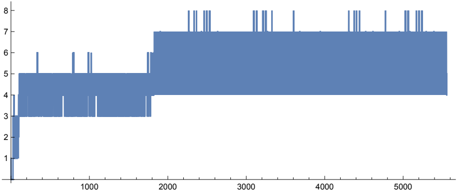
$\mathbf{w}_{5/4}$ — first 2000 rows



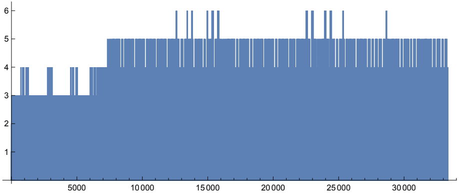
Large-scale structure

Write $\mathbf{w}_{5/4} = w(0)w(1)\cdots$.

$w(72i + 31)_{i \geq 0}$:



$w(i)_{i \geq 0}$:



Implied relationship:

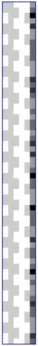
$$w(6i + 123061) = w(i + 5920) + \begin{cases} 3 & \text{if } i \equiv 0, 2 \mod 8 \\ 1 & \text{if } i \equiv 4, 6 \mod 8 \\ 2 & \text{if } i \equiv 1 \mod 2 \end{cases}$$

Characterization of $\mathbf{w}_{5/4}$

Theorem (Rowland–Stipulanti)

There exist words p, z of lengths $|p| = 6764$ and $|z| = 20226$ such that $\mathbf{w}_{5/4} = p\,\tau(\varphi(z)\varphi^2(z)\cdots)$.

z is a word on $\Sigma_8 = \{n_j : n \in \mathbb{Z}, 0 \leq j \leq 7\}$. z contains -1_0 and -1_2.



$$
\begin{aligned}
\varphi(n_0) &= 0_0 1_1 0_2 0_3 1_4 (n+3)_5\\
\varphi(n_1) &= 1_6 1_7 0_0 0_1 0_2 (n+2)_3\\
\varphi(n_2) &= 1_4 1_5 1_6 0_7 0_0 (n+3)_1\\
\varphi(n_3) &= 0_2 1_3 1_4 0_5 1_6 (n+2)_7\\
\varphi(n_4) &= 0_0 1_1 0_2 0_3 1_4 (n+1)_5\\
\varphi(n_5) &= 1_6 1_7 0_0 0_1 0_2 (n+2)_3\\
\varphi(n_6) &= 1_4 1_5 1_6 0_7 0_0 (n+1)_1\\
\varphi(n_7) &= 0_2 1_3 1_4 0_5 1_6 (n+2)_7
\end{aligned}
$$

Proof outline

1. Show that $p\,\tau(\varphi(z)\varphi^2(z)\cdots)$ avoids $\frac{5}{4}$-powers.
2. Show that decreasing any letter in $p\,\tau(\varphi(z)\varphi^2(z)\cdots)$ introduces a $\frac{5}{4}$-power ending at that position.

For previously studied words $\mathbf{w}_{a/b}$, Step 1 involves showing that φ is $\frac{a}{b}$-power-free. That is, if w is $\frac{a}{b}$-power-free then $\varphi(w)$ is $\frac{a}{b}$-power-free.

However, the morphism for $\mathbf{w}_{5/4}$ is not $\frac{5}{4}$-power-free:
For $n, m \in \mathbb{Z}$, the word $0_4 n_5 m_6$ is $\frac{5}{4}$-power-free, but its image is not:

$$\varphi(0_4 n_5 m_6) = 0_0 1_1 0_2 0_3 1_4 1_5 \;\; 1_6 1_7 0_0 0_1 0_2 (n+2)_3 \;\; 1_4 1_5 1_6 0_7 0_0 (m+1)_1$$

Pre-$\frac{5}{4}$-power-freeness

A word is pre-$\frac{5}{4}$-power-free if every factor xyx' with $|x| = \frac{1}{3}|y| = |x'|$ satisfies $\varphi(x) \neq \varphi(x')$.

$0_0 n_1 n_2 n_3 2_4$ is not pre-$\frac{5}{4}$-power-free because its image is a $\frac{5}{4}$-power:

$$\varphi(0_0 n_1 n_2 n_3 2_4) = 0_0 1_1 0_2 0_3 1_4 3_5 \;\; \varphi(n_1 n_2 n_3) \;\; 0_0 1_1 0_2 0_3 1_4 3_5$$

If w is pre-$\frac{5}{4}$-power-free, then w is $\frac{5}{4}$-power-free.

Proposition

Let Γ be the set

$$\{-3_0, -3_2, -2_0, -2_1, -2_2, -2_3, -2_5, -2_7, -1_1, -1_3, -1_4, -1_5, -1_6, -1_7, 0_4, 0_6\}.$$

If $w \in (\Sigma_8 \setminus \Gamma)^*$ is pre-$\frac{5}{4}$-power-free, then $\varphi(w)$ is pre-$\frac{5}{4}$-power-free.

Proof strategy

1. Sequence of results for establishing $\frac{5}{4}$-power-freeness:

   $z\varphi(z)\varphi^2(z)\cdots$ is pre-$\frac{5}{4}$-power-free.

   $\varphi(z\varphi(z)\varphi^2(z)\cdots)$ is $\frac{5}{4}$-power-free.

   $\tau(\varphi(z)\varphi^2(z)\cdots)$ is $\frac{5}{4}$-power-free.

   $p\tau(\varphi(z)\varphi^2(z)\cdots)$ is $\frac{5}{4}$-power-free.

2. For establishing lexicographic leastness:

   Case analysis and complicated induction.

Both steps involve large finite checks carried out programmatically.

References

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