The Sinkhorn limit of a matrix

Eric Rowland, joint work with **Jason Wu**

Mathematics Seminar
Hofstra University, 2024–2–21

2×2 matrix:

$$\begin{bmatrix} 6 & 4 \\ 1 & 6 \end{bmatrix}$$

Scale rows:

$$\begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{1}{7} & \frac{6}{7} \end{bmatrix} \approx \begin{bmatrix} .6 & .4 \\ .142857 & .857143 \end{bmatrix}$$

Scale columns:

$$\begin{bmatrix} .807692 & .318182 \\ .192308 & .681818 \end{bmatrix}$$

Iterate...

$$\begin{bmatrix} .750001 & .250001 \\ .249999 & .749999 \end{bmatrix}$$

$\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$ is the Sinkhorn limit of $\begin{bmatrix} 6 & 4 \\ 1 & 6 \end{bmatrix}$. Row and column sums are 1.

Sinkhorn 1964: The limit exists.

Another 2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Alternately scale rows and columns (double speed) ...

$$\begin{bmatrix} .585786 & .414214 \\ .414214 & .585786 \end{bmatrix} \approx \begin{bmatrix} 2 - \sqrt{2} & -1 + \sqrt{2} \\ -1 + \sqrt{2} & 2 - \sqrt{2} \end{bmatrix}$$

Fixed point!

A third 2×2 matrix:

$$\begin{bmatrix} 4 & 3 \\ 8 & 1 \end{bmatrix}$$

Alternately scale rows and columns...

$$\begin{bmatrix} .289898 & .710102 \\ .710102 & .289898 \end{bmatrix} \approx \begin{bmatrix} \frac{-1+\sqrt{6}}{5} & \frac{6-\sqrt{6}}{5} \\ \frac{6-\sqrt{6}}{5} & \frac{-1+\sqrt{6}}{5} \end{bmatrix}$$

Fixed point!

Theorem (Nathanson 2020)

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with positive entries,

$$\operatorname{Sink}(A) = \frac{1}{\sqrt{ad} + \sqrt{bc}} \begin{bmatrix} \sqrt{ad} & \sqrt{bc} \\ \sqrt{bc} & \sqrt{ad} \end{bmatrix}.$$

$$\operatorname{Sink}\left(\begin{bmatrix} 4 & 3 \\ 8 & 1 \end{bmatrix}\right) = \frac{1}{1+\sqrt{6}} \begin{bmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 1 \end{bmatrix} = \frac{-1+\sqrt{6}}{5} \begin{bmatrix} 1 & \sqrt{6} \\ \sqrt{6} & 1 \end{bmatrix}$$

For 3×3 matrices, the Sinkhorn limit was not known!

$$\operatorname{Sink}\left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix}\right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

What are these numbers? Assume they're algebraic.

For 2×2, the top left entry satisfies $(ad - bc)x^2 - 2adx + ad = 0$.

Compute the top left entry to high precision:

$x \approx .25033837405936848945454728685142925283386722173530167719 94$

Use the PSLQ integer relation algorithm to guess a polynomial:

(partial sums, LQ decomposition)

$$236379x^6 + 502124x^5 - 1610856x^4 + 19808x^3 + 661120x^2 - 94592x - 12288 = 0$$

Conjecture (Kevin Chen and Abel Varghese 2019, HUSSRP)

For 3×3 matrices A, the entries of $\operatorname{Sink}(A)$ have degree at most 6.

For a symmetric matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$:

Theorem (Ekhad–Zeilberger 2019)

The top left entry x of $\operatorname{Sink}(A)$ *satisfies* $c_4 x^4 + \cdots + c_1 x + c_0 = 0$, *where*

$$
\begin{aligned}
c_4 &= -(a_{12}^2 - a_{11}a_{22})(a_{13}^2 - a_{11}a_{33})(-a_{11}a_{22}a_{33} + a_{11}a_{23}^2 + a_{12}^2a_{33} - 2a_{12}a_{13}a_{23} + a_{13}^2a_{22}) \\
c_3 &= (-4a_{11}^3a_{22}^2a_{33}^2 + 4a_{11}^3a_{22}a_{23}^2a_{33} + 4a_{11}^2a_{12}^2a_{22}a_{33}^2 - 3a_{11}^2a_{12}^2a_{23}^2a_{33} - 2a_{11}^2a_{12}a_{13}a_{22}a_{23}a_{33} + 4a_{11}^2a_{13}^2a_{22}^2a_{33} \\
&\quad - 3a_{11}^2a_{13}^2a_{22}a_{23}^2 - 2a_{11}a_{12}^2a_{13}^2a_{22}a_{33} + 2a_{11}a_{12}^2a_{13}^2a_{23}^2 - a_{12}^4a_{13}^2a_{33} + 2a_{12}^3a_{13}^3a_{23} - a_{12}^2a_{13}^4a_{22}) \\
c_2 &= a_{11}(6a_{11}^2a_{22}^2a_{33}^2 - 6a_{11}^2a_{22}a_{23}^2a_{33} - 2a_{11}a_{12}^2a_{22}a_{33}^2 + 3a_{11}a_{12}^2a_{23}^2a_{33} - 2a_{11}a_{12}a_{13}a_{22}a_{23}a_{33} - 2a_{11}a_{13}^2a_{22}^2a_{33} \\
&\quad + 3a_{11}a_{13}^2a_{22}a_{23}^2 + 2a_{12}^3a_{13}a_{23}a_{33} - 3a_{12}^2a_{13}^2a_{22}a_{33} - a_{12}^2a_{13}^2a_{23}^2 + 2a_{12}a_{13}^3a_{22}a_{23}) \\
c_1 &= -a_{11}^2(4a_{11}a_{22}^2a_{33}^2 - 4a_{11}a_{22}a_{23}^2a_{33} + a_{12}^2a_{23}^2a_{33} - 2a_{12}a_{13}a_{22}a_{23}a_{33} + a_{13}^2a_{22}a_{23}^2) \\
c_0 &= a_{11}^3a_{22}a_{33}(a_{22}a_{33} - a_{23}^2)
\end{aligned}
$$

Computed with Gröbner bases.

For symmetric A, the limit $\mathrm{Sink}(A)$ requires more information!

If we know

$$\mathrm{Sink}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{ad}}{\sqrt{ad}+\sqrt{bc}} & ? \\ ? & ? \end{bmatrix}$$

then its bottom left entry is the top left entry of

$$\mathrm{Sink}\left(\begin{bmatrix} c & d \\ a & b \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{cb}}{\sqrt{cb}+\sqrt{da}} & \\ & \end{bmatrix}.$$

But if we only know

$$\mathrm{Sink}\left(\begin{bmatrix} a & b \\ b & d \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{ad}}{\sqrt{ad}+b} & ? \\ ? & ? \end{bmatrix}$$

then we can't determine its bottom left entry from

$$\mathrm{Sink}\left(\begin{bmatrix} b & d \\ a & b \end{bmatrix}\right).$$

For general A, it suffices to describe the top left entry of $\mathrm{Sink}(A)$.

What is it? System of equations...

Row scaling — multiplication on the left.
Column scaling — multiplication on the right.

$$\mathrm{Sink}(A) = RAC$$

$$\mathrm{Sink}(A) = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \quad R = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad C = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

9 equations from $\mathrm{Sink}(A) = RAC$:

$$\begin{aligned} s_{11} &= r_1 a_{11} c_1 & s_{12} &= r_1 a_{12} c_2 & s_{13} &= r_1 a_{13} c_3 \\ s_{21} &= r_2 a_{21} c_1 & s_{22} &= r_2 a_{22} c_2 & s_{23} &= r_2 a_{23} c_3 \\ s_{31} &= r_3 a_{31} c_1 & s_{32} &= r_3 a_{32} c_2 & s_{33} &= r_3 a_{33} c_3 \end{aligned}$$

6 equations from row and column sums:

$$\begin{aligned} s_{11} + s_{12} + s_{13} &= 1 & s_{11} + s_{21} + s_{31} &= 1 \\ s_{21} + s_{22} + s_{23} &= 1 & s_{12} + s_{22} + s_{32} &= 1 \\ s_{31} + s_{32} + s_{33} &= 1 & s_{13} + s_{23} + s_{33} &= 1 \end{aligned}$$

15 equations in $9 + 3 + 9 + 3 = 24$ variables. Want s_{11} in terms of a_{ij}.
Symmetric A only uses 15 variables because we can require $R = C$.
Gröbner basis computation...

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$
\begin{aligned}
b_6 = {} & (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\
& \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\
b_5 = {} & -6a_{11}^5a_{22}^2a_{23}a_{32}a_{33}^2 + 6a_{11}^5a_{22}a_{23}^2a_{32}^2a_{33} + 8a_{11}^4a_{12}a_{21}a_{22}a_{23}a_{32}a_{33}^2 \\
& - 5a_{11}^4a_{12}a_{21}a_{23}^2a_{32}^2a_{33} + 5a_{11}^4a_{12}a_{22}^2a_{23}a_{31}a_{33}^2 - 8a_{11}^4a_{12}a_{22}a_{23}^2a_{31}a_{32}a_{33} \\
& + 5a_{11}^4a_{13}a_{21}a_{22}^2a_{32}a_{33}^2 - 8a_{11}^4a_{13}a_{21}a_{22}a_{23}a_{32}^2a_{33} + 8a_{11}^4a_{13}a_{22}^2a_{23}a_{31}a_{32}a_{33} \\
& - 5a_{11}^4a_{13}a_{22}a_{23}^2a_{31}a_{32}^2 - 2a_{11}^3a_{12}^2a_{21}^2a_{23}a_{32}a_{33}^2 - 6a_{11}^3a_{12}^2a_{21}a_{22}a_{23}a_{31}a_{33}^2 \\
& + 6a_{11}^3a_{12}^2a_{21}a_{23}^2a_{31}a_{32}a_{33} + 2a_{11}^3a_{12}^2a_{22}a_{23}^2a_{31}^2a_{33} - 6a_{11}^3a_{12}a_{13}a_{21}^2a_{22}a_{32}a_{33}^2 \\
& + 6a_{11}^3a_{12}a_{13}a_{21}^2a_{23}a_{32}^2a_{33} - 4a_{11}^3a_{12}a_{13}a_{21}a_{22}^2a_{31}a_{33}^2 + 4a_{11}^3a_{12}a_{13}a_{21}a_{23}^2a_{31}a_{32}^2 \\
& - 6a_{11}^3a_{12}a_{13}a_{22}^2a_{23}a_{31}^2a_{33} + 6a_{11}^3a_{12}a_{13}a_{22}a_{23}^2a_{31}^2a_{32} + 2a_{11}^3a_{13}^2a_{21}^2a_{22}a_{32}^2a_{33} \\
& - 6a_{11}^3a_{13}^2a_{21}a_{22}^2a_{31}a_{32}a_{33} + 6a_{11}^3a_{13}^2a_{21}a_{22}a_{23}a_{31}a_{32}^2 - 2a_{11}^3a_{13}^2a_{22}^2a_{23}a_{31}^2a_{32} \\
& + a_{11}^2a_{12}^3a_{21}^2a_{23}a_{31}a_{33}^2 - a_{11}^2a_{12}^3a_{21}a_{23}^2a_{31}^2a_{33} + a_{11}^2a_{12}^2a_{13}a_{21}^3a_{32}a_{33}^2 \\
& + 4a_{11}^2a_{12}^2a_{13}a_{21}^2a_{22}a_{31}a_{33}^2 - 4a_{11}^2a_{12}^2a_{13}a_{21}^2a_{23}a_{31}a_{32}a_{33} \\
& + 4a_{11}^2a_{12}^2a_{13}a_{21}a_{22}a_{23}a_{31}^2a_{33} - 4a_{11}^2a_{12}^2a_{13}a_{21}a_{23}^2a_{31}^2a_{32} - a_{11}^2a_{12}^2a_{13}a_{22}a_{23}^2a_{31}^3 \\
& - a_{11}^2a_{12}a_{13}^2a_{21}^3a_{32}^2a_{33} + 4a_{11}^2a_{12}a_{13}^2a_{21}^2a_{22}a_{31}a_{32}a_{33} - 4a_{11}^2a_{12}a_{13}^2a_{21}^2a_{23}a_{31}a_{32}^2 \\
& + 4a_{11}^2a_{12}a_{13}^2a_{21}a_{22}^2a_{31}^2a_{33} - 4a_{11}^2a_{12}a_{13}^2a_{21}a_{22}a_{23}a_{31}^2a_{32} + a_{11}^2a_{12}a_{13}^2a_{22}^2a_{23}a_{31}^3 \\
& - a_{11}^2a_{13}^3a_{21}^2a_{22}a_{31}a_{32}^2 + a_{11}^2a_{13}^3a_{21}a_{22}^2a_{31}^2a_{32} - 2a_{11}a_{12}^2a_{13}^2a_{21}^2a_{22}a_{31}^2a_{33} \\
& + 2a_{11}a_{12}^2a_{13}^2a_{21}^2a_{23}a_{31}^2a_{32} - a_{12}^3a_{13}^2a_{21}^3a_{31}^2a_{33} + a_{12}^3a_{13}^2a_{21}^2a_{23}a_{31}^3 \\
& + a_{12}^2a_{13}^3a_{21}^3a_{31}^2a_{32} - a_{12}^2a_{13}^3a_{21}^2a_{22}a_{31}^3
\end{aligned}
$$

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$
\begin{aligned}
b_4 = a_{11}\big(&15a_{11}^4 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^4 a_{22} a_{23}^2 a_{32}^2 a_{33} - 12a_{11}^3 a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&+ 10a_{11}^3 a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 10a_{11}^3 a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 12a_{11}^3 a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&- 10a_{11}^3 a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 12a_{11}^3 a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 12a_{11}^3 a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&+ 10a_{11}^3 a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{11}^2 a_{12}^2 a_{21}^2 a_{23} a_{32} a_{33}^2 + 6a_{11}^2 a_{12}^2 a_{21} a_{22} a_{23} a_{31} a_{33}^2 \\
&- 6a_{11}^2 a_{12}^2 a_{21} a_{23}^2 a_{31} a_{32} a_{33} - a_{11}^2 a_{12}^2 a_{22} a_{23}^2 a_{31}^2 a_{33} + 6a_{11}^2 a_{12} a_{13} a_{21}^2 a_{22} a_{32} a_{33}^2 \\
&- 6a_{11}^2 a_{12} a_{13} a_{21}^2 a_{23} a_{32}^2 a_{33} + 6a_{11}^2 a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - 6a_{11}^2 a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 \\
&+ 6a_{11}^2 a_{12} a_{13} a_{22}^2 a_{23} a_{31}^2 a_{33} - 6a_{11}^2 a_{12} a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} - a_{11}^2 a_{13}^2 a_{21}^2 a_{22} a_{32}^2 a_{33} \\
&+ 6a_{11}^2 a_{13}^2 a_{21} a_{22}^2 a_{31} a_{32} a_{33} - 6a_{11}^2 a_{13}^2 a_{21} a_{22} a_{23} a_{31} a_{32}^2 + a_{11}^2 a_{13}^2 a_{22}^2 a_{23} a_{31}^2 a_{32} \\
&- 2a_{11} a_{12}^2 a_{13} a_{21}^2 a_{22} a_{31} a_{33}^2 + 2a_{11} a_{12}^2 a_{13} a_{21} a_{23}^2 a_{31}^2 a_{32} + 2a_{11} a_{12} a_{13}^2 a_{21}^2 a_{23} a_{31} a_{32}^2 \\
&- 2a_{11} a_{12} a_{13}^2 a_{21} a_{22}^2 a_{31}^2 a_{33} - 3a_{12}^2 a_{13}^2 a_{21}^2 a_{22} a_{31}^2 a_{33} + 3a_{12}^2 a_{13}^2 a_{21}^2 a_{23} a_{31}^2 a_{32}\big) \\
b_3 = 2a_{11}^2\big(&-10a_{11}^3 a_{22}^2 a_{23} a_{32} a_{33}^2 + 10a_{11}^3 a_{22} a_{23}^2 a_{32}^2 a_{33} + 4a_{11}^2 a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&- 5a_{11}^2 a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} + 5a_{11}^2 a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 - 4a_{11}^2 a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&+ 5a_{11}^2 a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - 4a_{11}^2 a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} + 4a_{11}^2 a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&- 5a_{11}^2 a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 - a_{11} a_{12}^2 a_{21} a_{22} a_{23} a_{31} a_{33}^2 + a_{11} a_{12}^2 a_{21} a_{23}^2 a_{31} a_{32} a_{33} \\
&- a_{11} a_{12} a_{13} a_{21}^2 a_{22} a_{32} a_{33}^2 + a_{11} a_{12} a_{13} a_{21}^2 a_{23} a_{32}^2 a_{33} - 2a_{11} a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 \\
&+ 2a_{11} a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 - a_{11} a_{12} a_{13} a_{22}^2 a_{23} a_{31}^2 a_{33} + a_{11} a_{12} a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} \\
&- a_{11} a_{13}^2 a_{21} a_{22}^2 a_{31} a_{32} a_{33} + a_{11} a_{13}^2 a_{21} a_{22} a_{23} a_{31} a_{32}^2 + a_{12}^2 a_{13} a_{21}^2 a_{23} a_{31} a_{32} a_{33} \\
&- a_{12}^2 a_{13} a_{21} a_{22} a_{23} a_{31}^2 a_{33} - a_{12} a_{13}^2 a_{21}^2 a_{22} a_{31} a_{32} a_{33} + a_{12} a_{13}^2 a_{21} a_{22} a_{23} a_{31}^2 a_{32}\big)
\end{aligned}
$$

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$
\begin{aligned}
b_2 &= a_{11}^3 \left(15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \right.\\
&\quad + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&\quad - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&\quad \left. + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 \right) \\
b_1 &= a_{11}^4 \left(-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \right.\\
&\quad \left. + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 \right) \\
b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} \left(a_{22} a_{33} - a_{23} a_{32}\right)
\end{aligned}
$$

Better formulation?

$$
\begin{aligned}
b_6 &= \left(a_{11} a_{22} - a_{12} a_{21}\right)\left(a_{11} a_{23} - a_{13} a_{21}\right)\left(a_{11} a_{32} - a_{12} a_{31}\right)\left(a_{11} a_{33} - a_{13} a_{31}\right) \\
&\quad \cdot \left(a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}\right) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}
$$

Multiply each b_i by a_{11}.

$a_{11} b_6$ is the product of 6 determinants involving a_{11} and 0 not.

$a_{11} b_0$ is mainly the product of 0 determinants involving a_{11} and 6 not.

Is $a_{11} b_i$ made of products of i determinants involving a_{11} and $6 - i$ not?

Let $R \subseteq \{2,3\}$ and $C \subseteq \{2,3\}$ with $|R| = |C|$. Define

$$\Delta\binom{R}{C} = \det A_{\{1\}\cup R,\{1\}\cup C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C} \qquad\qquad M(S) = \prod_{(R,C)\in S} \Delta\binom{R}{C} \cdot \prod_{(R,C)\notin S} \Gamma\binom{R}{C}$$

Coefficients:

$$\begin{aligned}
a_{11} b_6 &= a_{11} (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \Delta\binom{\{\}}{\{\}} \Delta\binom{\{2\}}{\{2\}} \Delta\binom{\{2\}}{\{3\}} \Delta\binom{\{3\}}{\{2\}} \Delta\binom{\{3\}}{\{3\}} \Delta\binom{\{2,3\}}{\{2,3\}} \\
&= M\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
a_{11} b_0 &= a_{11}^6 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32}) \\
&= \Gamma\binom{\{\}}{\{\}} \Gamma\binom{\{2\}}{\{2\}} \Gamma\binom{\{2\}}{\{3\}} \Gamma\binom{\{3\}}{\{2\}} \Gamma\binom{\{3\}}{\{3\}} \Gamma\binom{\{2,3\}}{\{2,3\}} \\
&= M\big(\big)
\end{aligned}$$

$$\begin{aligned}
a_{11} b_1 &= a_{11}^5 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
&= -3M\binom{\{\}}{\{\}} - M\binom{\{2\}}{\{2\}} - M\binom{\{2\}}{\{3\}} - M\binom{\{3\}}{\{2\}} - M\binom{\{3\}}{\{3\}} + M\binom{\{2,3\}}{\{2,3\}} \\
&= -3\Sigma\binom{\{\}}{\{\}} - \Sigma\binom{\{2\}}{\{2\}} + \Sigma\binom{\{2,3\}}{\{2,3\}}
\end{aligned}$$

$$\Sigma(S) = \sum_{T \equiv S} M(T)$$

Theorem (Rowland–Wu 2024)

Let A be a 3×3 matrix with positive entries.
The top left entry x of $\mathrm{Sink}(A)$ *satisfies* $d_6 x^6 + \cdots + d_1 x + d_0 = 0$, *where*

$$d_6 = \Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}\ \{3\}\ \{2,3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}\ \{3\}\ \{2,3\}}$$

$$d_5 = -3\Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}\ \{3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}\ \{3\}} - \Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}\ \{2,3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}\ \{2,3\}} + \Sigma\binom{\{2\}\ \{2\}\ \{3\}\ \{3\}\ \{2,3\}}{\{2\}\ \{3\}\ \{2\}\ \{3\}\ \{2,3\}}$$

$$d_4 = 4\Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}} + \Sigma\binom{\{\}\ \{2\}\ \{3\}\ \{2,3\}}{\{\}\ \{2\}\ \{3\}\ \{2,3\}} - 3\Sigma\binom{\{2\}\ \{2\}\ \{3\}\ \{3\}}{\{2\}\ \{3\}\ \{2\}\ \{3\}}$$

$$d_3 = -4\Sigma\binom{\{\}\ \{2\}\ \{2\}}{\{\}\ \{2\}\ \{3\}} - 5\Sigma\binom{\{\}\ \{2\}\ \{3\}}{\{\}\ \{2\}\ \{3\}} + \Sigma\binom{\{\}\ \{2\}\ \{2,3\}}{\{\}\ \{2\}\ \{2,3\}} + \Sigma\binom{\{2\}\ \{2\}\ \{3\}}{\{2\}\ \{3\}\ \{2\}} - \Sigma\binom{\{2\}\ \{3\}\ \{2,3\}}{\{2\}\ \{3\}\ \{2,3\}}$$

$$d_2 = 4\Sigma\binom{\{\}\ \{2\}}{\{\}\ \{2\}} - 3\Sigma\binom{\{\}\ \{2,3\}}{\{\}\ \{2,3\}} + \Sigma\binom{\{2\}\ \{3\}}{\{2\}\ \{3\}}$$

$$d_1 = -3\Sigma\binom{\{\}}{\{\}} - \Sigma\binom{\{2\}}{\{2\}} + \Sigma\binom{\{2,3\}}{\{2,3\}}$$

$$d_0 = \Sigma\big(\big).$$

The coefficients exhibit a surprising symmetry.

Why degree 6? $1 + 4 + 1 = 6$ determinants involve a_{11}: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

For $n \times n$ matrices: $\sum_{j=0}^{n-1} \binom{n-1}{j}^2 = \binom{2n-2}{n-1}$ determinants involve a_{11}.

Conjecture

For $n \times n$ matrices A, the entries of $\mathrm{Sink}(A)$ have degree at most $\binom{2n-2}{n-1}$.

2×2: degree $\binom{2}{1} = 2$ $\quad (ad - bc)x^2 - 2adx + ad = 0$

3×3: degree $\binom{4}{2} = 6$

4×4: degree $\binom{6}{3} = 20$ $\quad$ Gröbner basis computation is infeasible.

5×5: degree $\binom{8}{4} = 70$

Compute the Sinkhorn limit numerically to high precision.
Use PSLQ to guess a polynomial for the top left entry.
Do this many many times. . . for 1.5 CPU years.
Generalize to $m \times n$ matrices. Coefficients are simple functions of m, n!

Up to signs, we have a conjecture for the explicit polynomial for $\mathrm{Sink}(A)$.

References

Shalosh B. Ekhad and Doron Zeilberger, Answers to some questions about explicit Sinkhorn limits posed by Mel Nathanson, `https://arxiv.org/abs/1902.10783` (6 pages).

Melvyn B. Nathanson, Alternate minimization and doubly stochastic matrices, *Integers* **20A** (2020) Article #A10 (17 pages).

Richard Sinkhorn, A relationship between arbitrary positive matrices and doubly stochastic matrices, *The Annals of Mathematical Statistics* **35** (1964) 876–879.