Sequences arising from cellular automata

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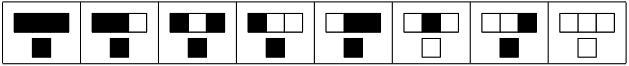
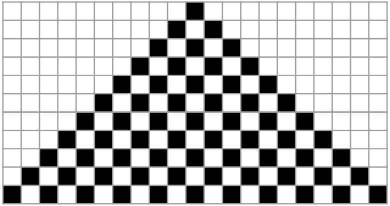
Outline

1. What is a cellular automaton?
2. Length of row n
3. The number of nonzero cells on row n
4. The cell on row n in column m

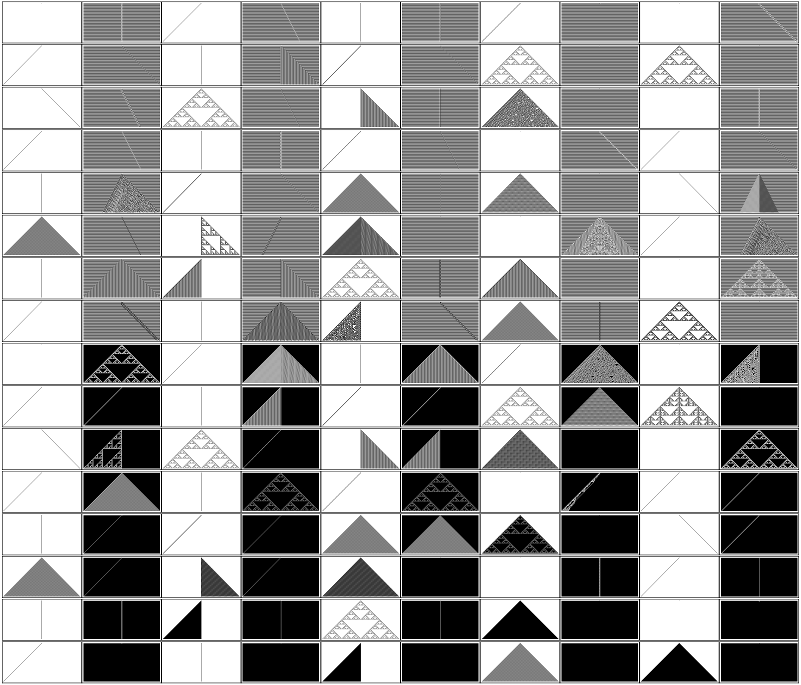
One-dimensional cellular automata

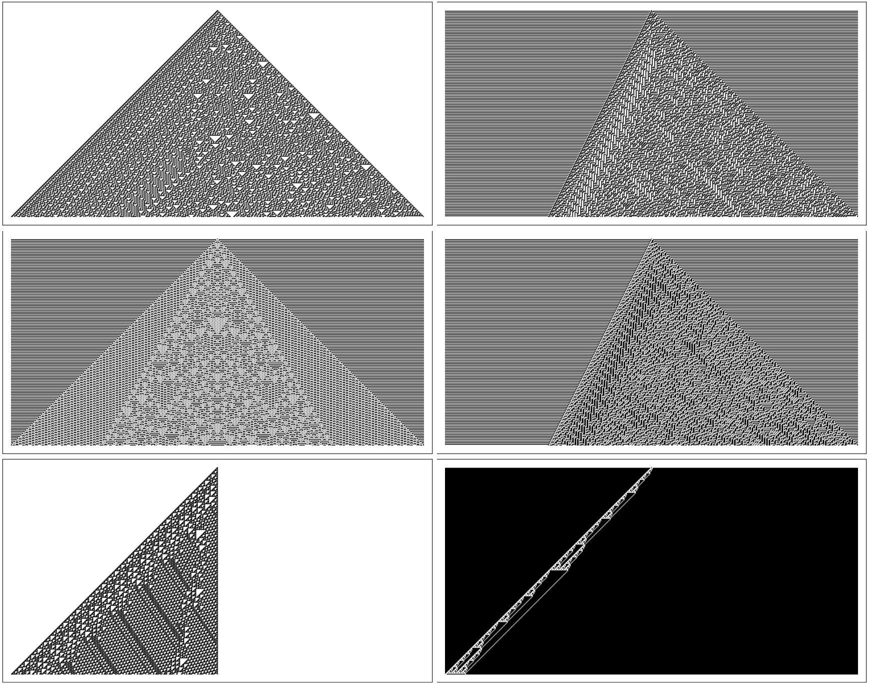
Let k and d be positive integers.
A one-dimensional, k-color cellular automaton depending on d cells consists of an alphabet Σ of size k, a function $i : \mathbb{Z} \to \Sigma$ (the initial condition), and a function $f : \Sigma^d \to \Sigma$ (the update rule).





Naming scheme: $11111010_2 = 250$.
Wolfram: Look at all k^{k^d} k-color rules depending on d cells.

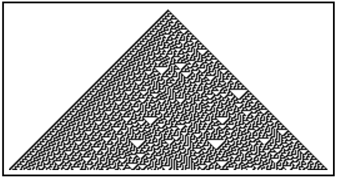




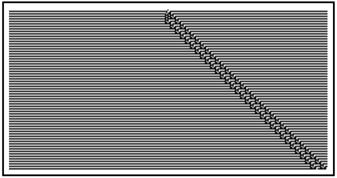
Behavior

The Goldilocks dilemma…

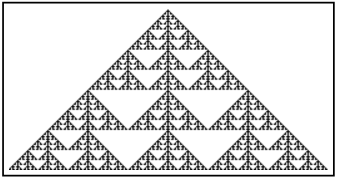
complex behavior —
difficult to treat mathematically



simple behavior — trivial



fractal behavior — just right



Sequences

Some sequences that arise from a cellular automaton:

- length of row n
  [Recent work with Charlie Brummitt.]
- the number of nonzero cells on row n
- the cell on row n in column m
- 2D sequence giving the entire evolution

We will mostly look at $k = 2$ colors.

Finiteness condition:
All but finitely many cells in the initial condition have the same color.

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Linear growth

For a given automaton, let $\ell(n)$ be number of cells on row n in the region that differs from the background.

A cellular automaton depending on d cells has maximal growth $\ell(n) \leq (d-1)n + c$.

Many length sequences grow linearly.

For $k = 2$ and $d \leq 3$, the only slopes that occur are $0, 1, 3/2, 2$.

What slopes occur for larger d?

Search

There are $2^{2^4} = 65536$ 2-color rules depending on $d = 4$ cells.

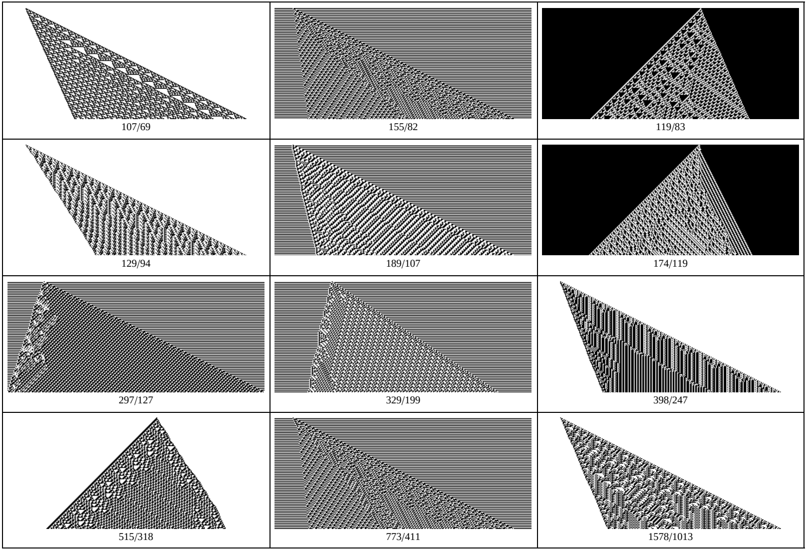
Look at each rule begun from a single black cell and a single white cell.

Detect growth sequences $\ell(n)$ that seem to be of the form

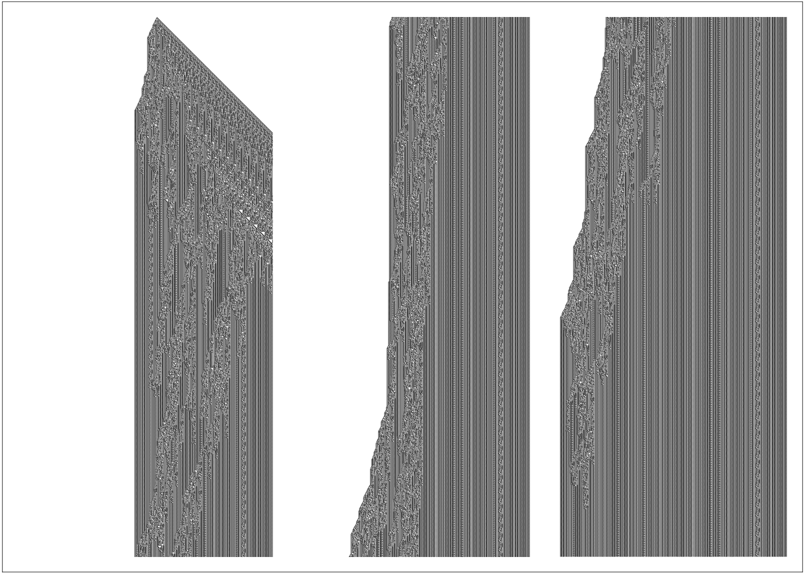
$$\ell(n) = \begin{cases} sn + c_0 & \text{if } n \equiv 0 \mod m \\ sn + c_1 & \text{if } n \equiv 1 \mod m \\ \quad\vdots & \qquad\vdots \\ sn + c_{m-1} & \text{if } n \equiv m-1 \mod m. \end{cases}$$

For the rest, attempt curve-fitting, and look at them manually.

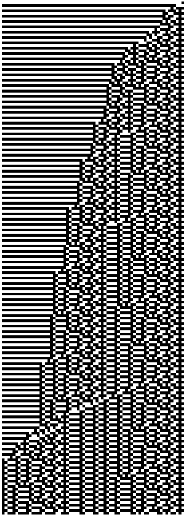
Search

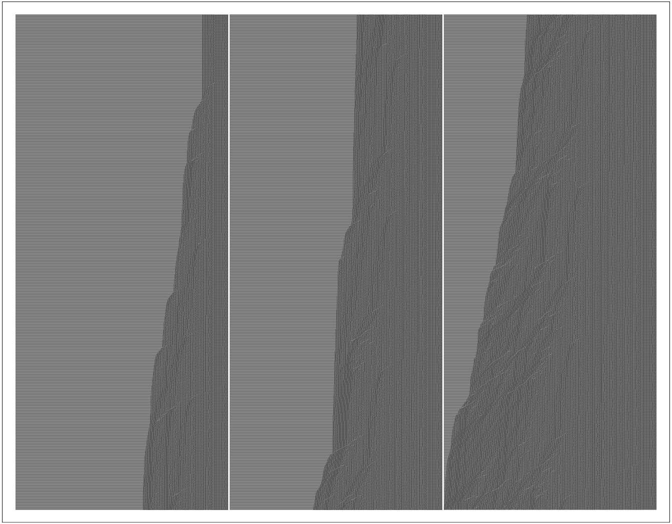

107/69
155/82
119/83
129/94
189/107
174/119
297/127
329/199
398/247
515/318
773/411
1578/1013

Misleading cases



Misleading cases



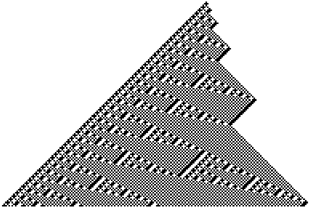


Around step 524500, growth increases rapidly (10000 steps shown).

Existence of slope?

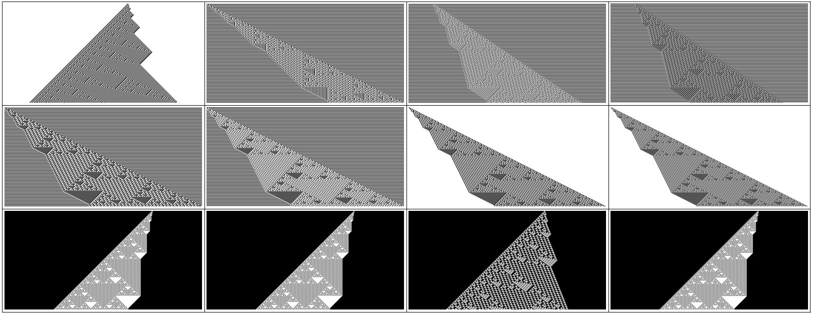
If $\ell(n) \in \Theta(n)$, does $\lim_{n\to\infty} \frac{\ell(n)}{n}$ necessarily exist?

No:



- $\liminf \ell(n)/n = 6/5$
- $\limsup \ell(n)/n = 3/2$

Boundaries given by morphisms

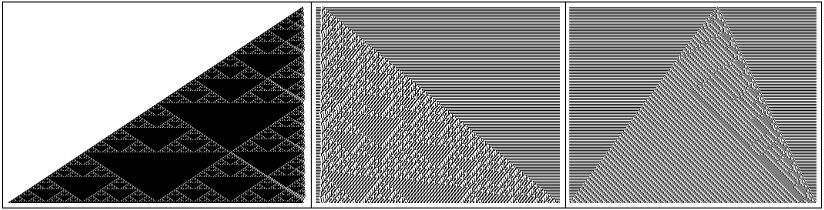


The difference sequence $\ell(n+1) - \ell(n)$ for each of these automata is the image, under some morphism, of the fixed point

$$\varphi^\omega(a) = abcbbbccbbbbbbccccbbbbbbbbbbbbcccccccc \cdots$$

of the morphism $\varphi(a) = abcb, \varphi(b) = bb, \varphi(c) = cc$.

Boundaries given by morphisms

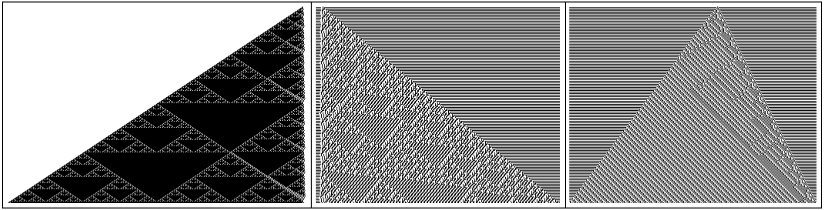


For the first automaton above, the difference sequence can be obtained by dropping the first two letters in the fixed point

$$\varphi^\omega(2) = 221221122122111221221122122111 \cdots$$

of the morphism $\varphi(1) = 1, \varphi(2) = 221$.

Boundaries given by morphisms



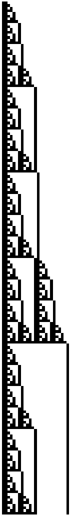
The difference sequence of the second is $(3\bar{1})^1(30)^2\psi(\varphi^\omega(a))$, where

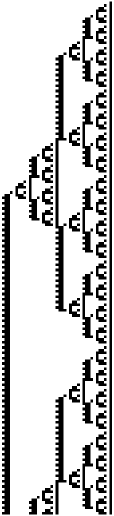
$$\varphi(a) = ac, \quad \varphi(b) = ad, \quad \varphi(c) = ba, \quad \varphi(d) = bb,$$
$$\psi(a) = (3\bar{1})^3(30)^2(3\bar{1})^3(30)^2(3\bar{1})^1(30)^2$$
$$\psi(b) = (3\bar{1})^3(30)^2(3\bar{1})^5(30)^2$$
$$\psi(c) = (3\bar{1})^5(30)^2(3\bar{1})^3(30)^2(3\bar{1})^1(30)^2$$
$$\psi(d) = (3\bar{1})^5(30)^2(3\bar{1})^5(30)^2.$$

Square-root growth

Some automata grow like $\sqrt{n}$:

- Rule 106 depending on 3 cells, begun from $\cdots$ □□□■■□□□ $\cdots$.
- Rule 39780 depending on 4 cells.

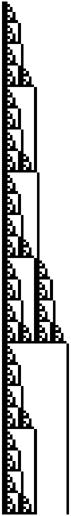




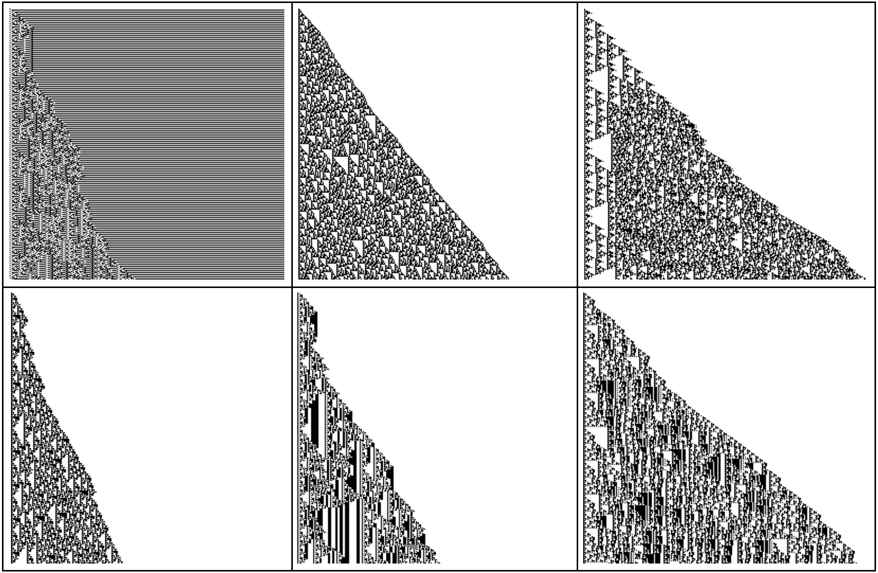
Row lengths in rule 106

The length $\ell(n)$ is (conjecturally) 2-regular:

$$\begin{aligned}
\ell(4n+1) &= 1/2\ell(4n) + 1/2\ell(4n+2) \\
\ell(8n+2) &= -2\ell(2n) + \ell(8n) + 2\ell(2n+1) \\
\ell(8n+3) &= -2\ell(2n) + \ell(8n) + 2\ell(2n+1) \\
\ell(8n+4) &= -3\ell(2n) + \ell(8n) + 3\ell(2n+1) \\
\ell(8n+6) &= -3\ell(2n) + \ell(8n) + 3\ell(2n+1) \\
\ell(8n+7) &= -4\ell(2n) + \ell(8n) + 4\ell(2n+1) \\
\ell(16n+0) &= -2\ell(n) + 3\ell(4n) + \ell(4n+2) - \ell(4n+3) \\
\ell(16n+8) &= -2\ell(n) + 1/2\ell(4n) + 7/2\ell(4n+2) - \ell(4n+3)
\end{aligned}$$



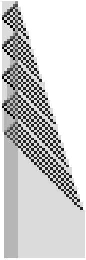
Chaotic boundaries



Existence of growth exponent?

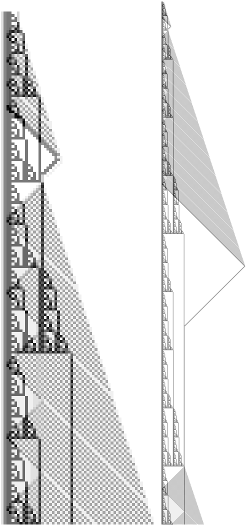
Does $\lim_{n\to\infty} \log_n \ell(n)$ necessarily exist?

No:
Graft a squaring automaton onto rule 106.



The result is an 18-color rule with $d = 4$.

- $\liminf \log_n \ell(n) = 1/2$
- $\limsup \log_n \ell(n) = 1$



Open questions

- Is there an automaton with $n^{1/3}$ growth?
- For which $0 \le \alpha \le 1$ do there exist automata with growth n^α?
- Is there an automaton that grows like $\sqrt{n} \log n$?

Slowest growth

If a cellular automaton is not eventually periodic, then it has at most $(k-1)^2 k^{\ell-1}$ rows of length ℓ.

So it grows at least logarithmically.

Open question

Does some cellular automaton grow logarithmically?

Binary representations of integers are not generated by any cellular automaton.

Neither are representations in the Gray code.

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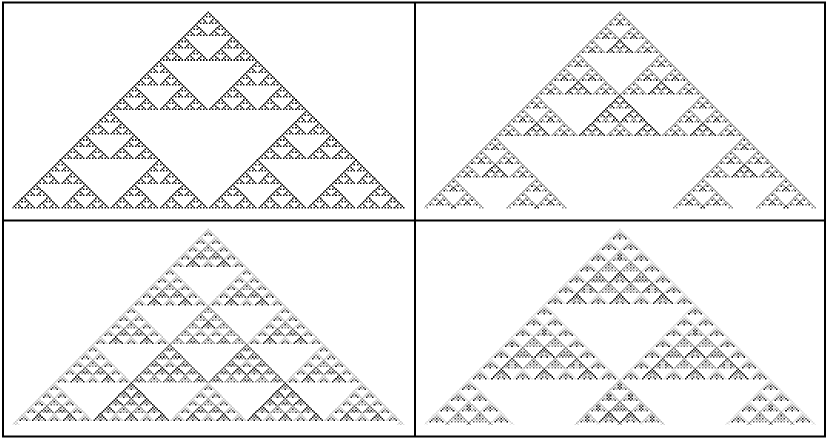
Binomial coefficients

Binomial coefficients modulo k are produced by cellular automata.



Nonzero binomial coefficients

Let $a_{p^\alpha}(n) = |\{0 \leq m \leq n : \binom{n}{m} \not\equiv 0 \mod p^\alpha\}|$.
Let $|n|_w$ be the number of occurrences of w in the base-p representation of n.

- Glaisher 1899:
$$a_2(n) = 2^{|n|_1}.$$
- Fine 1947:
$$a_p(n) = \prod_{i=0}^{l} (n_i + 1),$$
where $n = n_l \cdots n_1 n_0$ in base p.

For example, $a_5(n) = 2^{|n|_1} 3^{|n|_2} 4^{|n|_3} 5^{|n|_4}$.

It follows that $a_p(n)$ is p-regular.

Nonzero binomial coefficients

Rowland 2011:
Algorithm for obtaining a symbolic expression in $|n|_w$ for $a_{p^\alpha}(n)$.
It follows that $a_{p^\alpha}(n)$ is p-regular for each $\alpha \geq 0$.

For example:

$$a_{p^2}(n) = \left(\prod_{i=0}^{l} (n_i + 1) \right) \cdot \left(1 + \sum_{i=0}^{l-1} \frac{p - (n_i + 1)}{n_i + 1} \cdot \frac{n_{i+1}}{n_{i+1} + 1} \right).$$

Expressions for $p = 2$ and $p = 3$:

$$a_4(n) = 2^{|n|_1} \left(1 + \frac{1}{2} |n|_{10} \right)$$

$$a_9(n) = 2^{|n|_1} 3^{|n|_2} \left(1 + |n|_{10} + \frac{1}{4} |n|_{11} + \frac{4}{3} |n|_{20} + \frac{1}{3} |n|_{21} \right)$$

Nonzero binomial coefficients

Higher powers of 2:

$$a_8(n) = 2^{|n|_1} \left(1 + \frac{1}{8}|n|_{10}^2 + \frac{3}{8}|n|_{10} + |n|_{100} + \frac{1}{4}|n|_{110} \right)$$

$$\begin{aligned} \frac{a_{16}(n)}{2^{|n|_1}} = 1 &+ \frac{5}{12}|n|_{10} + \frac{1}{2}|n|_{100} + \frac{1}{8}|n|_{110} \\ &+ 2|n|_{1000} + \frac{1}{2}|n|_{1010} + \frac{1}{2}|n|_{1100} + \frac{1}{8}|n|_{1110} + \frac{1}{16}|n|_{10}^2 \\ &+ \frac{1}{2}|n|_{10}|n|_{100} + \frac{1}{8}|n|_{10}|n|_{110} + \frac{1}{48}|n|_{10}^3 \end{aligned}$$

Additive automata

More generally, we can add d cells modulo k.
And scale each entry by a constant.

Martin–Odlyzko–Wolfram 1984:
Encode the cells in row n as the coefficients of a polynomial $r_n(y)$.
Then such a rule corresponds to multiplication by a polynomial $q(y)$:
$r_{n+1}(y) = q(y) r_n(y)$.

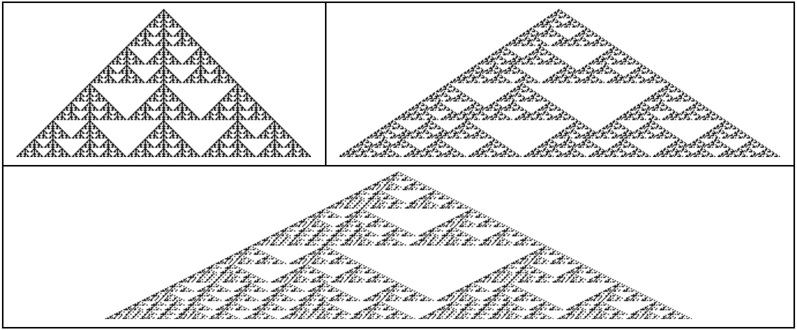
For example, binomial coefficients are the coefficients of $(1 + y)^n$.

The infinite evolution of the automaton is encoded in

$$\sum_{n \geq 0} r_n(y) x^n = \sum_{n \geq 0} q(y)^n r_0(y) x^n = \frac{r_0(y)}{1 - xq(y)}.$$

Additive automata

Here is $(1 + y + y^{d-1})^n$ over $\mathbb{F}_2$ for $d = 3, 4, 5$:

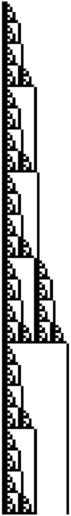


Amdeberhan–Stanley ~2008:
Let $f(x_1, \ldots, x_m) \in \mathbb{F}_{p^\alpha}[x_1, \ldots, x_m]$. The number $a(n)$ of nonzero terms in the expanded form of $f(x_1, \ldots, x_m)^n$ is p-regular.

Rule 106 again

The number $a(n)$ of black cells on row n is 2-regular:

$$a(4n+0) = a(n)$$
$$a(4n+1) = a(4n+2)$$
$$a(8n+2) = a(2n+1)$$
$$a(8n+3) = 2a(2n+1) - a(2n)$$
$$a(8n+6) = 2a(2n+1) - a(2n)$$
$$a(8n+7) = 4a(2n+1) - 3a(2n)$$



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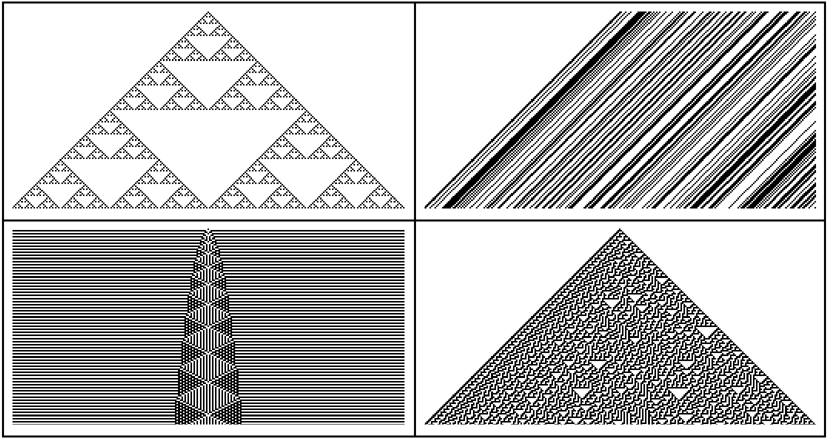
Column sequences

characteristic sequence of 2^n

bits of π

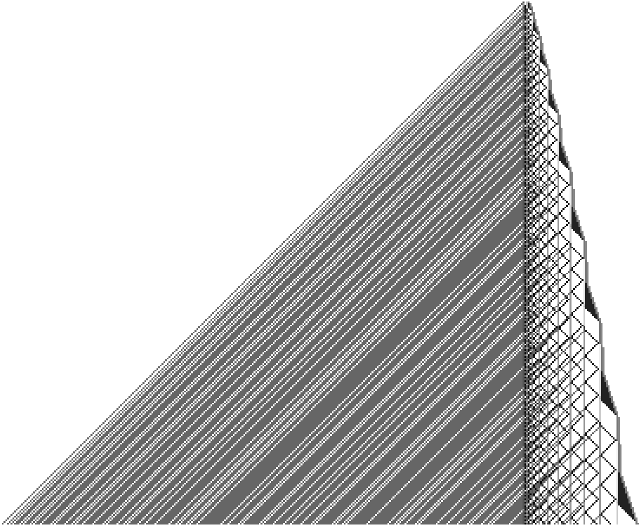


characteristic sequence of n^2

statistically random sequences

Characteristic sequence of primes

A 16-color rule depending on 3 cells that computes the primes:



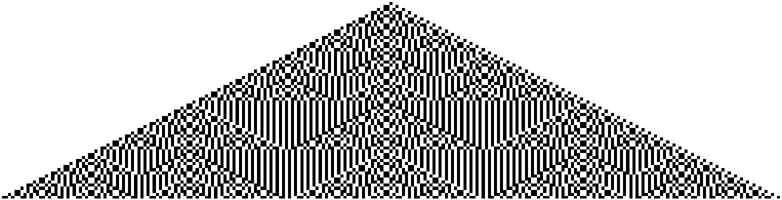
The Thue–Morse sequence

$$a(n) = \begin{cases} 0 & \text{if the binary representation of } n \text{ has an even number of 1s} \\ 1 & \text{if the binary representation of } n \text{ has an odd number of 1s.} \end{cases}$$

For $n \geq 0$, the Thue–Morse sequence is

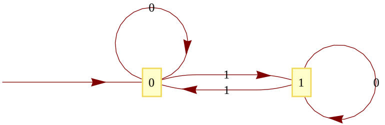
$$0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,1\,0\,0\,1\,0\,1\,1\,0\,0\,1\,1\,0\,1\,0\,0\,1\,\cdots.$$

It occurs as a column of this 2-color automaton depending on 5 cells:



The Thue–Morse sequence

The Thue–Morse sequence is 2-automatic:



The generating function $f(x) = \sum_{n \geq 0} a(n)x^n$ is algebraic over $\mathbb{F}_2(x)$:

$$(x+1)^3 f(x)^2 + (x^2+1)f(x) + x = 0.$$

Automatic sequences

Furstenberg 1967:
A power series $f(x)$ over $\mathbb{F}_{p^\alpha}$ is algebraic if and only if it is the diagonal of a rational series $g(x, y)$ over $\mathbb{F}_{p^\alpha}$.

Litow–Dumas 1993:
Write $g(x/y, y) = P(x, y)/Q(x, y) = \sum_{n\geq 0} r_n(y)x^n$.
Then $Q(x, y)$ encodes a linear recurrence satisfied by $r_n(y)$.
This gives a cellular automaton with memory.

If $a(n)$ is p-automatic, then there exists a cellular automaton with column $a(n)$.

Corollary:
Every periodic sequence occurs.

Open questions

- Does every periodic sequence on an alphabet of size k occur in a k-color cellular automaton?
- Does every k-automatic sequence occur in a cellular automaton (if k is not prime)?
- Does some nonperiodic 3-automatic sequence occur in a 2-color cellular automaton?
- Does every morphic sequence occur in a cellular automaton?
- Does the Fibonacci word

  $$abaababaabaababaababaabaababaabaab\cdots$$

  (the fixed point of $\varphi(a) = ab, \varphi(b) = a$) occur in a cellular automaton?
- Exhibit some sequence that does not occur as the column of a cellular automaton.