Powers of a sequence satisfying a recurrence

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Fibonacci sequence:
$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$

Definition:
$F(0) = 0$
$F(1) = 1$
$F(n+2) = F(n+1) + F(n)$ for all $n \geq 0$.

Question

Do powers of the Fibonacci sequence also satisfy recurrences?
Why would we even ask that question?

Fibonacci sequence:
$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \dots$

$F(n)^2$:
$0, 1, 1, 4, 9, 25, 64, 169, 441, 1156, 3025, 7921, 20736, 54289, 142129, \dots$

Can we guess a recurrence?

$$a\, F(n)^2 + b\, F(n+1)^2 = F(n+2)^2$$

$$\begin{aligned} n = 0: &\qquad 0a + 1b = 1 \\ n = 1: &\qquad 1a + 1b = 4 \\ n = 2: &\qquad 1a + 4b = 9 \\ &\qquad\qquad \vdots \end{aligned}$$

$b = 1$ and $a = 3$, but then $a + 4b = 3 + 4 = 7 \neq 9$. No solution!

$F(n)$:
$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \dots$

$F(n)^2$:
$0, 1, 1, 4, 9, 25, 64, 169, 441, 1156, 3025, 7921, 20736, 54289, 142129, \dots$

Can we guess a larger recurrence?

$$a\,F(n)^2 + b\,F(n+1)^2 + c\,F(n+2)^2 = F(n+3)^2$$

$$\begin{aligned} n = 0: &\qquad 0a + 1b + 1c = 4 \\ n = 1: &\qquad 1a + 1b + 4c = 9 \\ n = 2: &\qquad 1a + 4b + 9c = 25 \\ &\qquad\qquad \vdots \end{aligned}$$

$a = -1,\ b = 2,\ c = 2$
$s(n) = F(n)^2$ seems to satisfy $s(n+3) = 2s(n+2) + 2s(n+1) - s(n)$.

m	recurrence for $F(n)^m$	size
0	$s(n+1) = s(n)$	1
1	$s(n+2) = s(n+1) + s(n)$	2
2	$s(n+3) = 2s(n+2) + 2s(n+1) - s(n)$	3
3	$s(n+4) = 3s(n+3) + 6s(n+2) - 3s(n+1) - s(n)$	4
4	$s(n+5) = 5s(n+4) + 15s(n+3) - 15s(n+2) - 5s(n+1) + s(n)$	5

What are these coefficients?

More basic: Why does the recurrence for $F(n)^m$ have size $m+1$?

Even more basic: Why do powers of Fibonacci satisfy recurrences?

Fibonacci belongs to a class of sequences with nice closure properties.

Theorem

Let $s(n)_{n\geq 0}$ and $t(n)_{n\geq 0}$ be constant-recursive sequences. Then $(s(n)+t(n))_{n\geq 0}$ and $(s(n)t(n))_{n\geq 0}$ are also constant-recursive sequences.

The Hierarchy of Integer Sequences:
`https://ericrowland.github.io/THIS.html` (Chapter 15)

Moreover, suppose the recurrences for $s(n)_{n\geq 0}$, $t(n)_{n\geq 0}$ have sizes r_1, r_2.

$(s(n)+t(n))_{n\geq 0}$ satisfies a recurrence with size r_1+r_2.
$(s(n)t(n))_{n\geq 0}$ satisfies a recurrence with size $r_1 r_2$.

$\left(F(n)^2\right)_{n\geq 0}$ satisfies a recurrence with size 4.

Why does the minimal recurrence for $F(n)^2$ have size 3?

$F(n+2) - F(n+1) - F(n) = 0$
Characteristic polynomial: $x^2 - x - 1$
Roots: $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$

Binet's formula:

$$F(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Write $F(n) = c\phi^n + d\bar{\phi}^n$.

Then $F(n)^2 = c^2\phi^{2n} + 2cd(\phi\bar{\phi})^n + d^2\bar{\phi}^{2n}$
$= c^2\phi^{2n} + 2cd(-1)^n + d^2\bar{\phi}^{2n}$.

Polynomial: $(x+1)(x-\phi^2)(x-\bar{\phi}^2) = x^3 - 2x^2 - 2x + 1$.
Recurrence: $s(n+3) - 2s(n+2) - 2s(n+1) + s(n) = 0$

Why does the recurrence for $F(n)^m$ have size $m+1$?

Products of 2 elements of $\{\phi, \bar{\phi}\}$: $\phi^2,\ \phi\bar{\phi},\ \bar{\phi}^2$
Products of 3 elements of $\{\phi, \bar{\phi}\}$: $\phi^3,\ \phi^2\bar{\phi},\ \phi\bar{\phi}^2,\ \bar{\phi}^3$

$\vdots$

Products of m elements of $\{\phi, \bar{\phi}\}$: $\phi^m,\ \phi^{m-1}\bar{\phi},\ \dots,\ \phi\bar{\phi}^{m-1},\ \bar{\phi}^m$

$m+1$ products, and they're all distinct. ✓

General powers:

Theorem

If $m \geq 0$ and $s(n)_{n\geq 0}$ is a constant-recursive sequence with rank r, then $\left(s(n)^m\right)_{n\geq 0}$ is constant-recursive with rank $\leq \binom{m+r-1}{m}$.

$r = 2$: $\quad \binom{m+r-1}{m} = \binom{m+1}{m} = \frac{(m+1)!}{m!\,1!} = m+1$

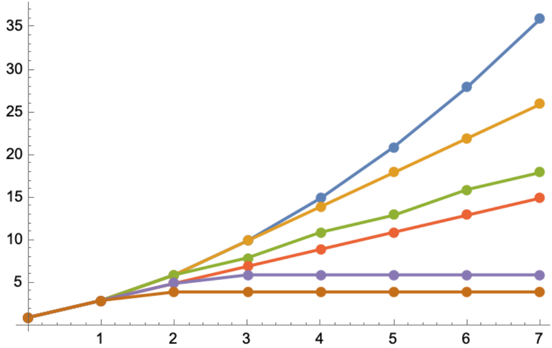
Powers of Fibonacci attain the upper bound!

Upper bound for $r = 3$: $\binom{m+r-1}{m} = \binom{m+2}{m} = \frac{(m+2)!}{m!\,2!} = \frac{(m+1)(m+2)}{2}$

$1, 3, 6, 10, 15, 21, 28, 36, \ldots$

Are there sequences that don't attain the upper bound?

recurrence	$\left(\operatorname{rank} \left(s(n)^m\right)_{n \geq 0}\right)_{m \geq 0}$
$s(n+3) = s(n+2) + s(n+1) + s(n)$	$1, 3, 6, 10, 15, 21, 28, 36, \dots$
$s(n+3) = 8s(n+2) - 8s(n+1) - 8s(n)$	$1, 3, 6, 10, 14, 18, 22, 26, \dots$
$s(n+3) = -5s(n+2) - 10s(n+1) - 12s(n)$	$1, 3, 6, 8, 11, 13, 16, 18, \dots$
$s(n+3) = -s(n+2) - 15s(n+1) - 15s(n)$	$1, 3, 5, 7, 9, 11, 13, 15, \dots$
$s(n+3) = -4s(n+2) - 8s(n+1) - 8s(n)$	$1, 3, 5, 6, 6, 6, 6, 6, \dots$
$s(n+3) = -2s(n+2) - 4s(n+1) - 8s(n)$	$1, 3, 4, 4, 4, 4, 4, 4, \dots$



At least 6 different behaviors for rank-3 sequences. How many are there?

Partial motivation for all of this...

In 1976, George Andrews rediscovered Ramanujan's "lost notebook".

$s(n)_{n\geq 0}$: $1, 135, 11161, 926271, 76869289, 6379224759, 529398785665, \ldots$
$t(n)_{n\geq 0}$: $2, 138, 11468, 951690, 78978818, 6554290188, 543927106802, \ldots$
$u(n)_{n\geq 0}$: $2, 172, 14258, 1183258, 98196140, 8149096378, 676276803218, \ldots$

Each sequence satisfies $s(n+3) = 82s(n+2) + 82s(n+1) - s(n)$.

$$\boxed{s(n)^3 + t(n)^3 = u(n)^3 + (-1)^n} \text{ for all } n \geq 0.$$

Characteristic polynomial: $x^3 - 82x^2 - 82x + 1$
Roots: -1, $\frac{83+9\sqrt{85}}{2}$, $\frac{83-9\sqrt{85}}{2}$ $\qquad \frac{83+9\sqrt{85}}{2} \cdot \frac{83-9\sqrt{85}}{2} = 1$

Size of the recurrence for $s(n)^m$: $1, 3, 5, 7, 9, 11, 13, 15, \ldots$ Not maximal!