Pattern avoidance in words

Julia Carrigan, Isaiah Hollars, Eric Rowland

$\frac{1}{3}$ **Mathematics Seminar**
Hofstra University, 2022–09–16

Definition

A word on a set Σ is a sequence of elements from Σ.

For this talk: $\Sigma = \{0, 1\}$ (the "alphabet")
Example: $w = 0110$

Pattern containment:

Example

0110 contains 11 (as a contiguous subword).
0110 avoids 00.

How many length-n words avoid 01?

length 0:	ϵ	1
length 1:	$0, 1$	2
length 2:	$00, 10, 11$	3
length 3:	$000, 100, 110, 111$	4
$\vdots$		$\vdots$
length n:		$n + 1$

How many length-n words avoid 10?

length 3:	$000, 001, 011, 111$	4

Also $n + 1$.

There is a bijection from words avoiding 01 to words avoiding 10.
(Actually two bijections!)

How many length-n words avoid 00?

length 0:	ϵ	1
length 1:	$0, 1$	2
length 2:	$01, 10, 11$	3
length 3:	$010, 011, 101, 110, 111$	5
length 4:	$0101, 0110, 0111, 1010, 1011, 1101, 1110, 1111$	8
$\vdots$		$\vdots$
length n:		$F(n+2)$

How many length-n words avoid 11? Also $F(n+2)$.

Definition

Two patterns are avoiding-equivalent if they are avoided by the same number of length-n words for all $n \geq 0$.

Length-2 patterns come in two equivalence classes: $\{01, 10\}$ and $\{00, 11\}$.

What are the equivalence classes of length-3 patterns?

pattern p	number of length-n words avoiding p for $n = 0, 1, \ldots$
000	$1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, \ldots$
001	$1, 2, 4, 7, 12, 20, 33, 54, 88, 143, 232, \ldots$
010	$1, 2, 4, 7, 12, 21, 37, 65, 114, 200, 351, \ldots$
011	$1, 2, 4, 7, 12, 20, 33, 54, 88, 143, 232, \ldots$
100	$1, 2, 4, 7, 12, 20, 33, 54, 88, 143, 232, \ldots$
101	$1, 2, 4, 7, 12, 21, 37, 65, 114, 200, 351, \ldots$
110	$1, 2, 4, 7, 12, 20, 33, 54, 88, 143, 232, \ldots$
111	$1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, \ldots$

Equivalence classes: $\{000, 111\}, \{010, 101\}, \{001, 011, 100, 110\}$

No surprises.

What are the equivalence classes of length-4 patterns?

equivalence class	self-overlap lengths
$\{0000, 1111\}$	$1, 2, 3, 4$
$\{0101, 1010\}$	$2, 4$
$\{0010, 0100, 0110, 1001, 1011, 1101\}$	$1, 4$
$\{0001, 0011, 0111, 1000, 1100, 1110\}$	4

Surprise!

Where does 0110 overlap itself?

```
   0110      0110      0110    0110
0110       0110      0110      0110
```

Theorem (Solovyov 1966, Guibas 1979)

If two patterns have the same set of self-overlap lengths, then they are avoiding-equivalent.

Proof: Generating series.

Is there a bijective proof?

Question

If two patterns p, q are avoiding-equivalent, is there a natural bijection from the length-n words avoiding p to the length-n words avoiding q?

length-4 words avoiding $p = 011$:

0000
0001 $\mapsto$ 0111
0010 $\mapsto$ 0110
0100
0101
1000
1001 $\mapsto$ 1011
1010
1100
1101
1110
1111

length-4 words avoiding $q = 001$:

0000
0100
0101
0110
0111
1000
1010
1011
1100
1101
1110
1111

Idea: Replace all instances of q with p. 0001 $\mapsto$ 0011 $\mapsto$ 0111

Let ϕ_L be the map that iteratively replaces the leftmost q with p.

Let $A_n(p)$ denote the set of length-n words that avoid p.

Lemma

If p, q are equal-length patterns and $w \in A_n(p)$, then $\phi_L(w) \in A_n(q)$.

A border of p is a subword that is both a prefix and a suffix of p.

Theorem

Let p, q be patterns such that every nontrivial border of p is also a border of q. Then ϕ_L is a bijection from $A_n(p)$ to $A_n(q)$.

$p = 011$ and $q = 001$ have no nontrivial borders. ✓

Non-example

$p = 0100$ and $q = 1011$ both have length-1 borders, but $0 \neq 1$.
$\phi_L(1011110) = 0100110$.
$\phi_L(0101011) = 0100100 = \phi_L(1011100)$. Not a bijection!