Morphic words governing the boundaries of cellular automata

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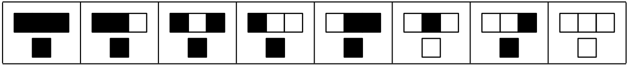
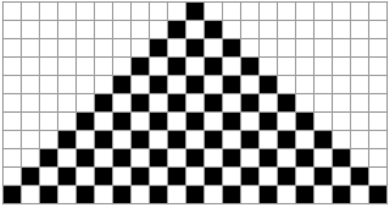
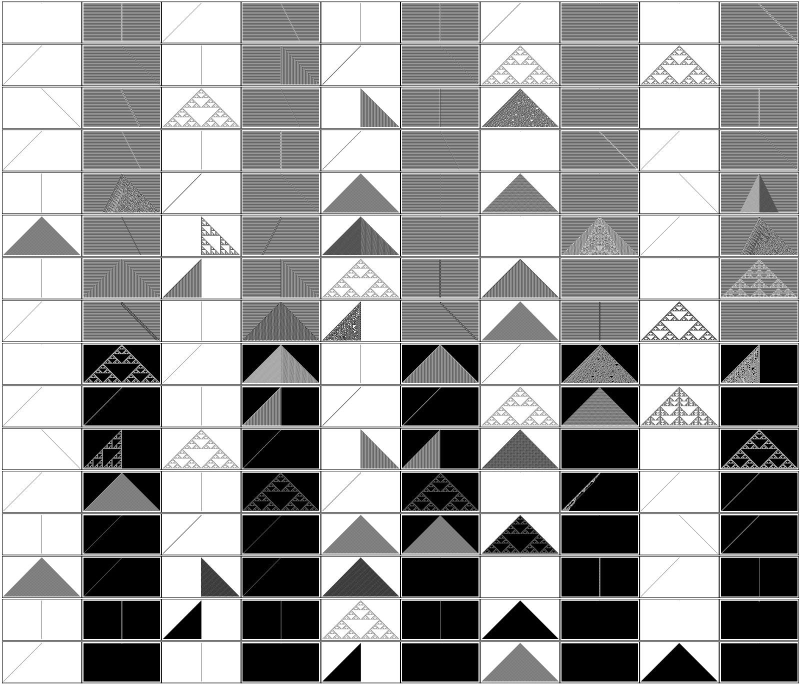
One-dimensional cellular automata

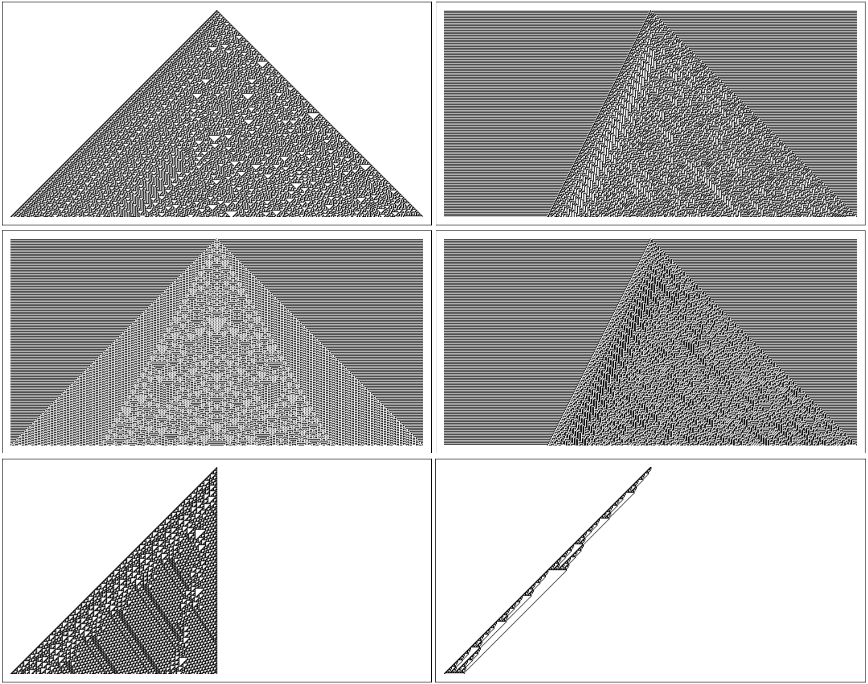
- alphabet Σ of size k (for example $\{0, 1, \ldots, k-1\}$)
- function $i : \mathbb{Z} \to \Sigma$ (the initial condition)
- function $f : \Sigma^d \to \Sigma$ (the update rule)





Naming scheme: $11111010_2 = 250$.
Wolfram: Look at all k^{k^d} k-color rules depending on d cells.

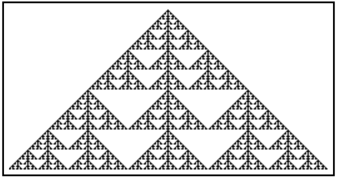




Row lengths

Definition

$\ell(n)$ = width of region on row n that differs from the background



For example, $\ell(n) = 2n + 1$.

Upper bound: $\ell(n) \leq (d - 1)n + c$.

For many automata, $\ell(n)$ grows linearly.

Linear growth

Growth of the form

$$\ell(n) = \begin{cases} an + c_0 & \text{if } n \equiv 0 \mod m \\ an + c_1 & \text{if } n \equiv 1 \mod m \\ \quad\vdots & \qquad\vdots \\ an + c_{m-1} & \text{if } n \equiv m-1 \mod m \end{cases}$$

for $n \geq n_{\min}$ (for some $m, n_{\min} \in \mathbb{Z}$ and $a, c_i \in \mathbb{Q}$)
can be detected by determining whether the difference sequence

$$\ell(n+1) - \ell(n)$$

is eventually periodic.

Automated–manual search

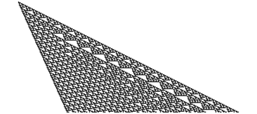
For $k = 2$ and $d \leq 3$, the only growth rates that occur are $0, 1, 3/2, 2$.

What happens for larger d?

There are $2^{2^4} = 65536$ 2-color rules depending on $d = 4$ cells.

- Consider each rule begun from $\cdots$ □□□■□□□ $\cdots$ and $\cdots$ ■■■□■■■ $\cdots$. There are 25088 equivalence classes.
- Filter out automata with sequences $\ell(n+1) - \ell(n)$ that seem to be eventually periodic.
- Examine the remaining automata manually for reducibility.
- For the remaining automata, fit curves to $\ell(n)$.

The most complex growth rates for $d = 4$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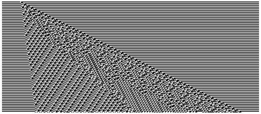
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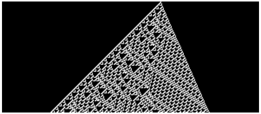
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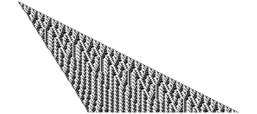
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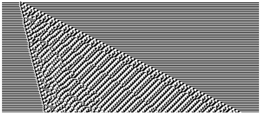
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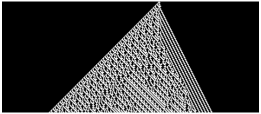
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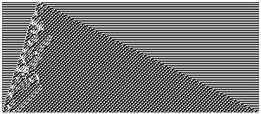
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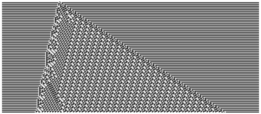
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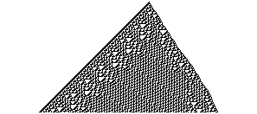
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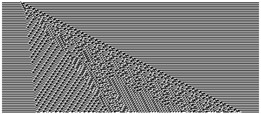
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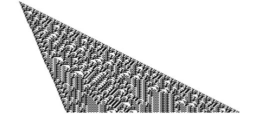
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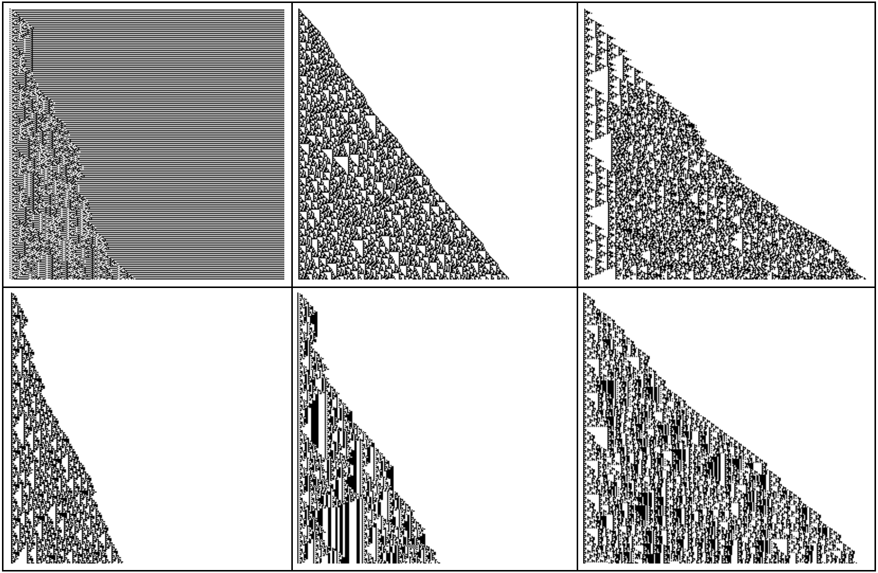
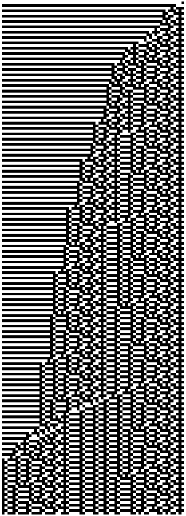
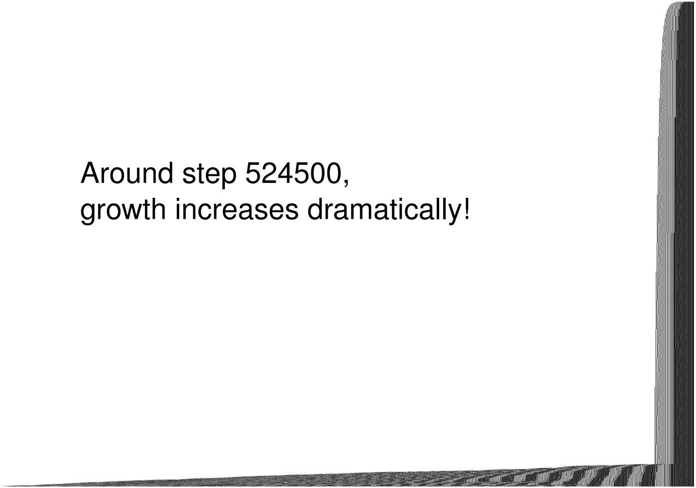
Chaotic boundaries



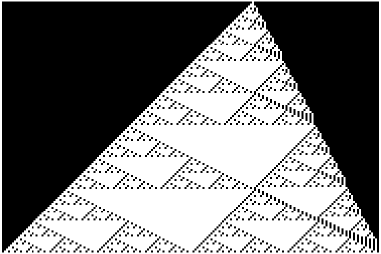
A misleading automaton



Around step 524500,
growth increases dramatically!



Nonperiodic linear growth



The difference sequence $\ell(n+1) - \ell(n)$ is

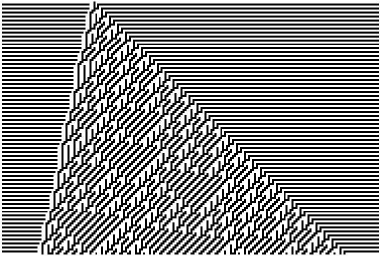
$$1, 2, 2, 1, 1, 2, 2, 1, 2, 2, 1, 1, 1, 2, 2, 1, 2, 2, 1, 1, 2, 2, 1, 2, 2, 1, 1, 1, 1, \dots.$$

It can be obtained by dropping the first two letters of the fixed point

$$\varphi^\omega(2) = 2212211221221112212211221221111\cdots$$

of the morphism $\varphi(1) = 1, \varphi(2) = 221$.

More nonperiodic linear growth



The difference sequence is $(3\bar{1})^1(30)^2\psi(\varphi^\omega(A))$, where

$$\varphi(A) = AC, \quad \varphi(B) = AD, \quad \varphi(C) = BA, \quad \varphi(D) = BB,$$
$$\psi(A) = (3\bar{1})^3(30)^2(3\bar{1})^3(30)^2(3\bar{1})^1(30)^2$$
$$\psi(B) = (3\bar{1})^3(30)^2(3\bar{1})^5(30)^2$$
$$\psi(C) = (3\bar{1})^5(30)^2(3\bar{1})^3(30)^2(3\bar{1})^1(30)^2$$
$$\psi(D) = (3\bar{1})^5(30)^2(3\bar{1})^5(30)^2.$$

Boundary word

Definition

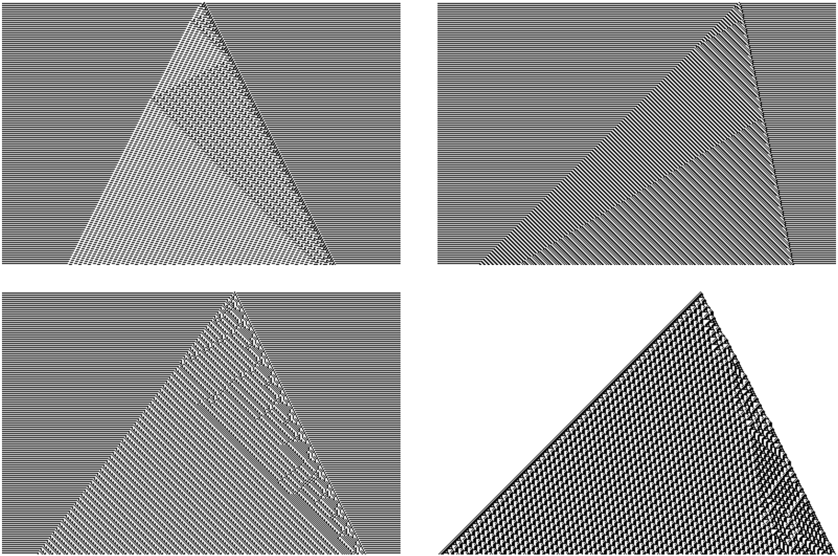
The boundary word is the infinite word $(\ell(n+1) - \ell(n))_{n \geq 0}$.

The boundary word is not necessarily a word on a finite subset of $\mathbb{Z}$.
But often it is.

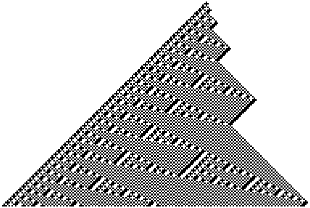
A classification of the 25088 automata is as follows.

- 24287 have eventually periodic boundary words.
- 81 have morphic boundary words that are not eventually periodic.
- 720 have boundaries that do not appear to be morphic.

Oscillating particles



Linear growth without a growth rate



Growth is linear: $\ell(n) \in \Theta(n)$.

But $\lim_{n\to\infty} \dfrac{\ell(n)}{n}$ does not exist.

- $\liminf \ell(n)/n = 6/5$
- $\limsup \ell(n)/n = 3/2$

Boundary word is $\psi(\varphi^\omega(A))$, where

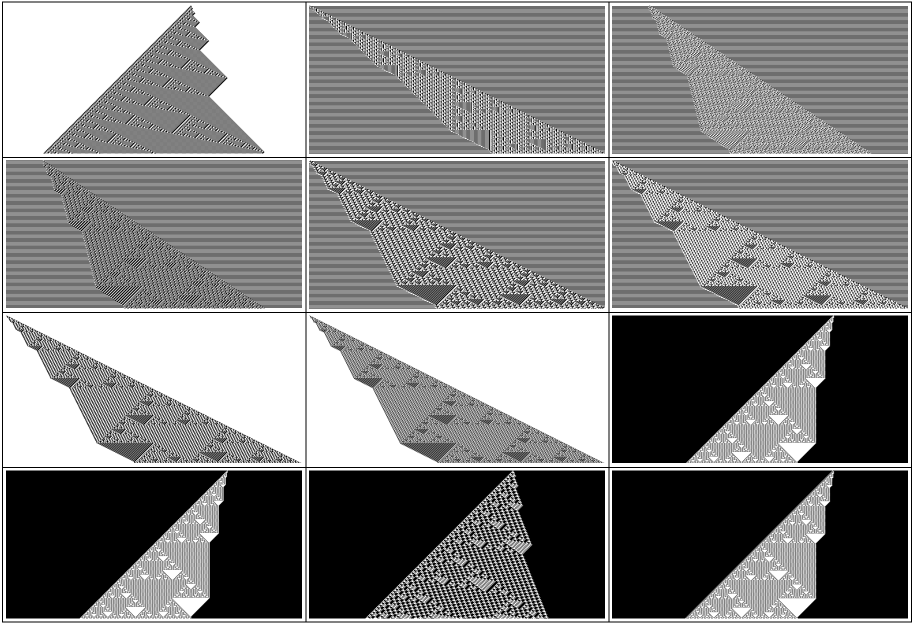
$$\varphi : A \to ABCB,\ B \to BB,\ C \to CC$$
$$\psi : A \to \epsilon,\ B \to 2,\ C \to 0.$$

The fixed point of φ is

$$\varphi^\omega(A) = ABCBBBCCBBBBBBCCCCBBBBBBBBBBBB\cdots.$$

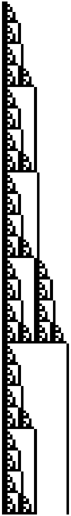
The frequencies of B and C don't exist!

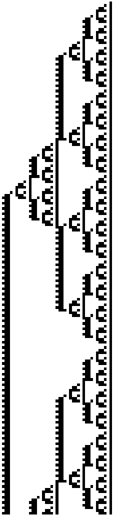
Automata with the same morphism φ



Some automata don't grow linearly:

- Rule 106 depending on 3 cells, begun from ··· □□□■■□□□ ··· .
- Rule 39780 depending on 4 cells.





$d = 3$ rule 106

The boundary word for rule 106 is

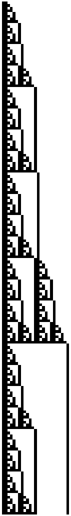
$$\psi(\varphi^\omega(A)) = 1101001100000000100000000011010011\cdots,$$

where

$$\varphi : A \to ABCD,\ B \to CCAB,\ C \to CCCC,\ D \to CCCD$$
$$\psi : A \to 1,\ B \to 1,\ C \to 0,\ D \to 1$$

Square-root growth rate can be derived from φ.

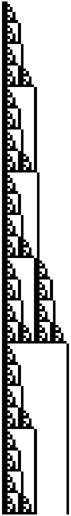
φ is 4-uniform.



Row lengths in rule 106

The length $\ell(n)$ is 2-regular:

$$\begin{aligned}
\ell(4n+1) &= 1/2\ell(4n) + 1/2\ell(4n+2) \\
\ell(8n+2) &= -2\ell(2n) + \ell(8n) + 2\ell(2n+1) \\
\ell(8n+3) &= -2\ell(2n) + \ell(8n) + 2\ell(2n+1) \\
\ell(8n+4) &= -3\ell(2n) + \ell(8n) + 3\ell(2n+1) \\
\ell(8n+6) &= -3\ell(2n) + \ell(8n) + 3\ell(2n+1) \\
\ell(8n+7) &= -4\ell(2n) + \ell(8n) + 4\ell(2n+1) \\
\ell(16n+0) &= -2\ell(n) + 3\ell(4n) + \ell(4n+2) - \ell(4n+3) \\
\ell(16n+8) &= -2\ell(n) + 1/2\ell(4n) + 7/2\ell(4n+2) - \ell(4n+3)
\end{aligned}$$



$d = 4$ rule 39780

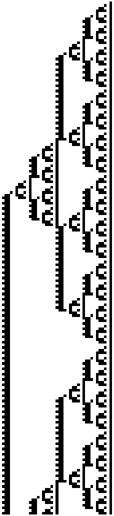
Rule 39780 also grows like $\sqrt{n}$.

Its boundary word is $\psi(\varphi^{\omega}(A))$, where

$$\varphi : A \to ABC,\ B \to DAB,$$
$$C \to CECE,\ D \to CECD,\ E \to CECE$$
$$\psi : A \to 2,\ B \to 2,\ C \to 1,\ D \to 0,\ E \to -1$$

φ is not uniform.

$\ell(n)$ is evidently not 2-regular.



Final thoughts

Moral of the story

Properties of the automaton are reflected in the boundary word.

Vague principal:
Sufficiently simple boundaries are characterized by morphic words.