IntegerSequences: a package for computing with k-regular sequences

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IntegerSequences is a *Mathematica* package for computing with integer sequences. Available from
`https://people.hofstra.edu/Eric_Rowland/packages.html`

There is particular emphasis on k-regular sequences, which arise in combinatorics, number theory, and computer science.

Today's demo:

- Guessing sequences from finitely many terms.
- Computing closure properties.
- Computing automata for a sequence modulo p^α.

Constant-recursive sequences

Definition

$s(n)_{n\geq 0}$ is constant-recursive if

$$\left\{s(n+i)_{n\geq 0} : i \geq 0\right\}$$

is contained in a finite-dimensional vector space.

Example (Fibonacci sequence)

$F(n+0)_{n\geq 0} = 0, 1, 1, 2, 3, 5, \ldots$
$F(n+1)_{n\geq 0} = 1, 1, 2, 3, 5, 8, \ldots$
$F(n+2)_{n\geq 0} = 1, 2, 3, 5, 8, 13, \ldots = F(n)_{n\geq 0} + F(n+1)_{n\geq 0}$

So each shift $F(n+i)_{n\geq 0}$ is an element of $\langle F(n)_{n\geq 0}, F(n+1)_{n\geq 0}\rangle$.

k-regular sequences

Definition (Allouche–Shallit 1992)

Let $k \geq 2$.
A sequence $s(n)_{n \geq 0}$ is k-constant-recursive or k-regular if

$$\left\{ s(k^e n + i)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq i \leq k^e - 1 \right\}$$

is contained in a finite-dimensional vector space.

Example (ruler sequence)

$k = 2$

$$s(1n+0)_{n \geq 0} = 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, \dots$$
$$s(2n+0)_{n \geq 0} = 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \dots$$
$$s(2n+1)_{n \geq 0} = 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, \dots$$
$$s(4n+1)_{n \geq 0} = 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \dots = s(2n+1)_{n \geq 0} - s(n)_{n \geq 0}$$
$$s(4n+3)_{n \geq 0} = 2, 3, 2, 4, 2, 3, 2, 5, 2, 3, \dots = 2s(2n+1)_{n \geq 0} - s(n)_{n \geq 0}$$

So each shift $s(2^e n + i)_{n \geq 0}$ is an element of $\langle s(n)_{n \geq 0}, s(2n+1)_{n \geq 0} \rangle$.

Matrix characterization

$s(n)_{n \geq 0}$ is constant-recursive if and only if
$s(n) = u\, M^n\, v$ for some matrix M and vectors u, v:

$$\begin{bmatrix} F(n+1) \\ F(n+2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F(n) \\ F(n+1) \end{bmatrix} \quad \Longrightarrow \quad F(n) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Matrix characterization

Let $n_\ell \cdots n_1 n_0$ be the base-k representation of n.
$s(n)_{n \geq 0}$ is k-regular if and only if $s(n) = u\, M(n_0)\, M(n_1) \cdots M(n_\ell)\, v$ for some matrices $M(0), \ldots, M(k-1)$ and vectors u, v:

$$\begin{bmatrix} s(2n+1) \\ s(4n+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} s(n) \\ s(2n+1) \end{bmatrix}; \quad \begin{bmatrix} s(2n+2) \\ s(4n+3) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} s(n) \\ s(2n+1) \end{bmatrix}$$

$\Longrightarrow$

$$u = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad M(0) = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \quad M(1) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example

$n = 11 = 1011_2$:

$$\begin{aligned} s(11) &= u\, M(1)\, M(1)\, M(0)\, M(1)\, v \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= 2 \end{aligned}$$

Representation

IntegerSequences uses the matrices $M(d)$ and the vectors u, v to represent a k-regular sequence.

demo...