# Growth of one-dimensional cellular automata

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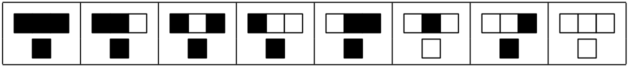
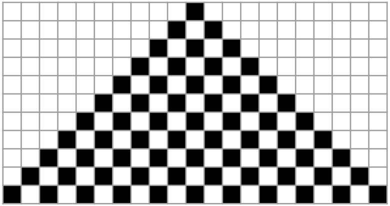
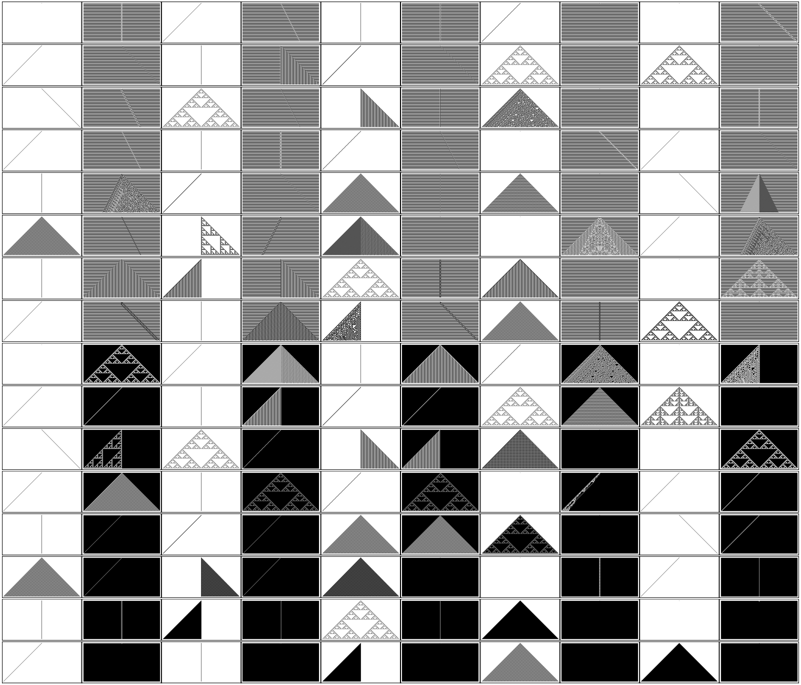
# Outline

- alphabet $\Sigma$ of size $k$ (for example $\{0, 1, \ldots, k-1\}$)
- function $i : \mathbb{Z} \to \Sigma$ (the initial condition)
- function $f : \Sigma^d \to \Sigma$ (the update rule)





Wolfram: Look at all $k^{k^d}$ $k$-color rules depending on $d$ cells.

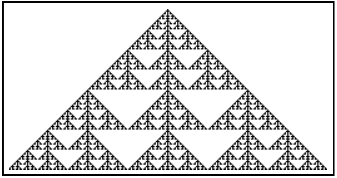


# Outline

1. What is a cellular automaton?
2. Growth rates
3. Growth exponents

# Row lengths

$\ell(n) =$ width of region on row $n$ that differs from the background



For example, $\ell(n) = 2n + 1$.

Upper bound: $\ell(n) \leq (d - 1)n + c$.

For many automata, $\ell(n)$ grows linearly.
In particular, if $\ell(n + 1) - \ell(n)$ is eventually periodic.

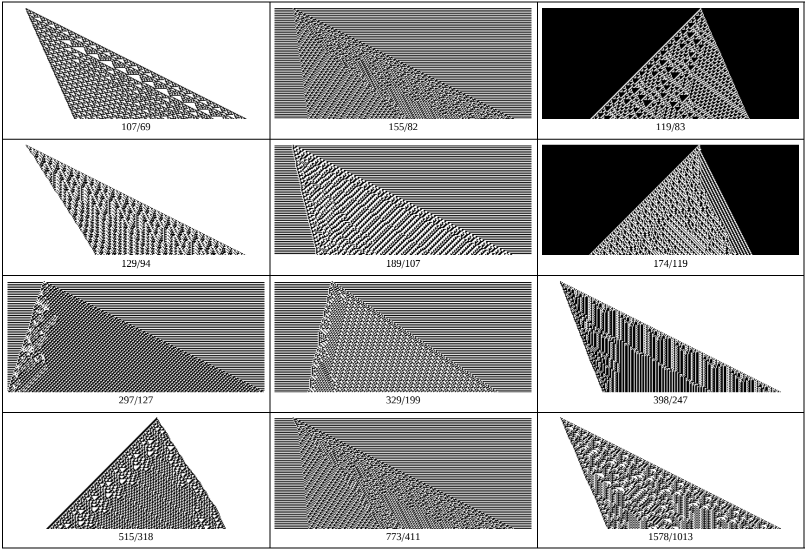
# Automated–manual search

For $k = 2$ and $d \leq 3$, the only growth rates are $0, 1, 3/2, 2$.

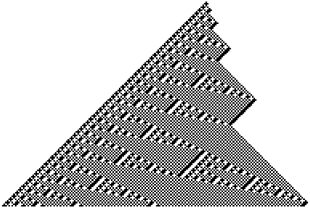
What happens for larger $d$?

There are $2^{2^4} = 65536$ 2-color rules depending on $d = 4$ cells.

- Consider each rule begun from $\cdots$ □□□■□□□ $\cdots$ and $\cdots$ ■■■□■■■ $\cdots$.
- Filter out automata with sequences $\ell(n+1) - \ell(n)$ that seem to be eventually periodic.
- Examine the remaining automata manually for reducibility.
- For the remaining automata, fit curves to $\ell(n)$.


107/69
155/82
119/83
129/94
189/107
174/119
297/127
329/199
398/247
515/318
773/411
1578/1013

# Existence of growth rate

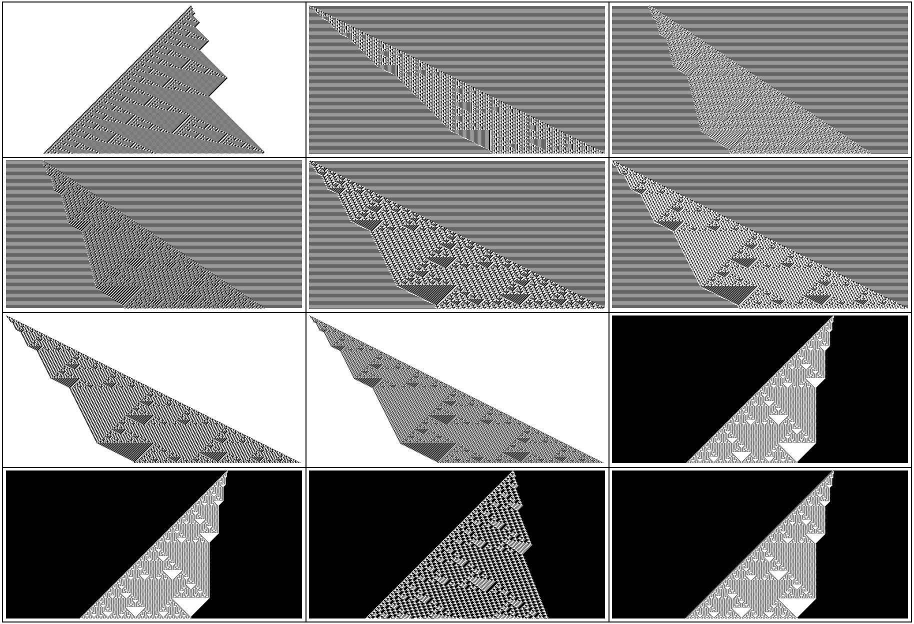


Growth is linear:
$c_1 n < \ell(n) < c_2 n$

But $\lim\limits_{n\to\infty} \dfrac{\ell(n)}{n}$ does not exist.

- $\liminf \ell(n)/n = 6/5$
- $\limsup \ell(n)/n = 3/2$

# Automata with the same boundary structure

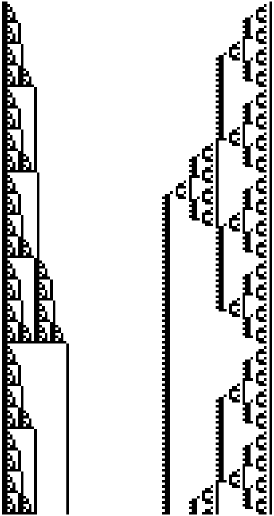


# Outline

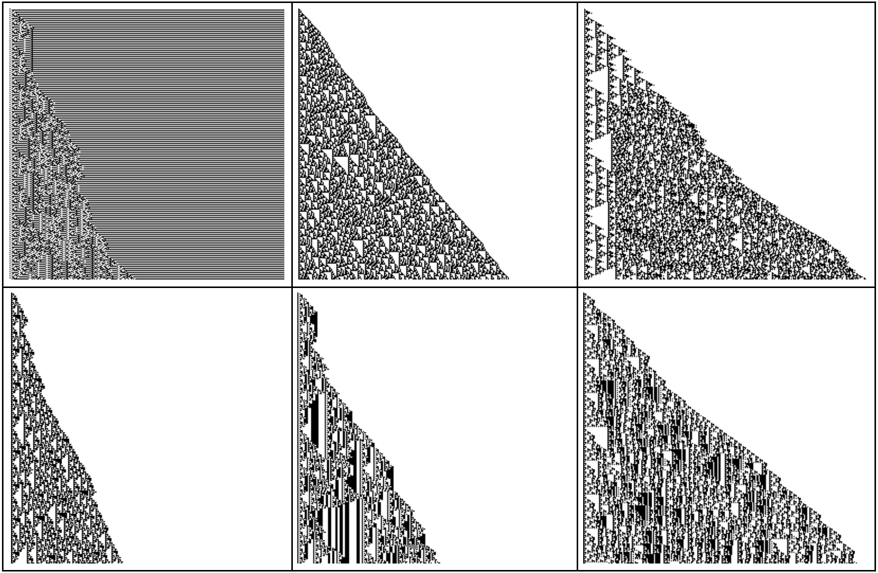
# Square-root growth

Some automata grow like $\sqrt{n}$:

- Rule 106 depending on 3 cells.
- Rule 39780 depending on 4 cells.



# Chaotic boundaries



# Data

Data we can store for $\ell(n)$:

- exact growth rate and period length (for reducible linear growth)
- approximate growth rate (for irreducible linear growth)
- morphism (for fractal boundaries)
- approximate growth exponent

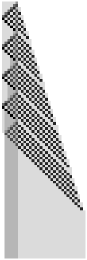
What do we do with this cellular automaton data?

Make it programmatically accessible in a *Mathematica* package.

# Existence of growth exponent

Does $\lim_{n\to\infty} \log_n \ell(n)$ necessarily exist? No!

Graft a squaring automaton onto rule 106.



The result is an 18-color rule with $d = 4$.

- $\liminf \log_n \ell(n) = 1/2$
- $\limsup \log_n \ell(n) = 1$