Formulas for Primes

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The sequence of primes

$\cancel{1}\ 2\ 3\ \cancel{4}\ 5\ \cancel{6}\ 7\ \cancel{8}\ \cancel{9}\ \cancel{10}\ 11\ \cancel{12}\ 13\ \cancel{14}\ \cancel{15}\ \cancel{16}\ 17\ \cancel{18}\ 19\ \cancel{20}\ \cancel{21}\ \cancel{22}\ 23\ \cancel{24}\ \cancel{25} \cdots$

The sieve of Eratosthenes generates the sequence of primes:

$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, \dots$

Two questions:

- Is it easy to tell when a number is prime? (1 slide)
- Are there formulas that easily produce primes? (24 slides)

Can primality be determined quickly?

Trial division: Test divisibility by all numbers $2 \leq m \leq \sqrt{n}$.

Wilson's Theorem (Lagrange 1773)

If $n \geq 2$, then n is prime if and only if n divides $(n-1)! + 1$.

For example, 5 divides $4! + 1 = 25$, but 6 doesn't divide $5! + 1 = 121$.

But determining whether an ℓ-digit number is prime using Wilson's theorem requires multiplying $\approx 10^\ell$ numbers.

Is there a polynomial-time algorithm for testing primality? Yes.

Agrawal, Kayal, & Saxena (2002) provided an algorithm that determines whether an ℓ-digit number is prime in $c \cdot \ell^{12}$ steps.

Willans' formula for the nth prime (1964)

$$p_n = 1 + \sum_{i=1}^{2^n} \left\lfloor \left(\frac{n}{\sum_{j=1}^{i} \left\lfloor \left(\cos \frac{(j-1)!+1}{j}\pi \right)^2 \right\rfloor} \right)^{1/n} \right\rfloor$$

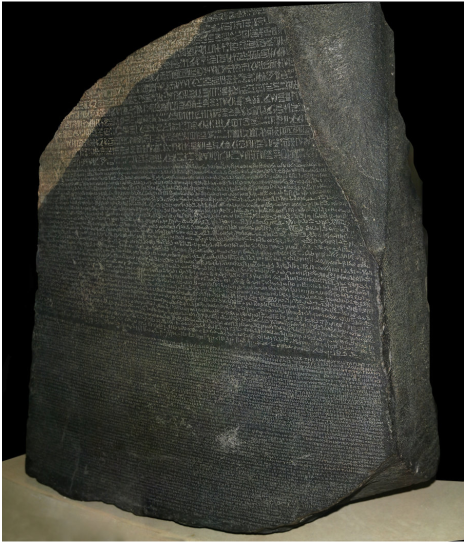
$$\frac{(j-1)!+1}{j} = \begin{cases} \text{an integer} & \text{if } j = 1 \text{ or } j \text{ is prime} \\ \text{not an integer} & \text{if } j \geq 2 \text{ and } j \text{ is not prime} \end{cases}$$

$$\left\lfloor \left(\cos \frac{(j-1)!+1}{j}\pi \right)^2 \right\rfloor = \begin{cases} 1 & \text{if } j = 1 \text{ or } j \text{ is prime} \\ 0 & \text{if } j \geq 2 \text{ and } j \text{ is not prime} \end{cases}$$

$$\sum_{j=1}^{i} \left\lfloor \left(\cos \frac{(j-1)!+1}{j}\pi \right)^2 \right\rfloor = (\#\text{ primes} \leq i) + 1$$

$$\left\lfloor \left(\frac{n}{(\#\text{ primes} \leq i) + 1} \right)^{1/n} \right\rfloor = \begin{cases} 1 & \text{if } i < p_n \\ 0 & \text{if } i \geq p_n \end{cases}$$

Translation



Fermat primes

n	0	1	2	3	4	5	6	7	8	9	10
2^n+1	2	3	5	9	17	33	65	129	257	513	1025

If 2^n+1 is prime, must n be a power of 2 (or $n=0$)? Yes:

$$a^m - b^m = (a-b)\cdot\left(a^{m-1} + a^{m-2}b + a^{m-3}b^2 + \cdots + b^{m-1}\right)$$

Suppose n is divisible by some odd m.
Let $a = 2^{n/m}$ and $b = -1$:
$a^m - b^m = (2^{n/m})^m - (-1)^m = 2^n + 1$ is divisible by $a - b = 2^{n/m} + 1$.

$2^{16} + 1 = 65537$ is also prime!
Fermat conjectured that $2^{2^k} + 1$ is prime for all $k \geq 0$.

But Euler factored $2^{32} + 1 = 4294967297 = 641 \times 6700417$.
And no Fermat primes have been found since!

Mersenne primes

n	0	1	2	3	4	5	6	7	8	9	10
$2^n - 1$	0	1	3	7	15	31	63	127	255	511	1023

Marin Mersenne (1588–1648)

If $2^n - 1$ is prime, must n be prime? Yes:

$$2^{km} - 1 = (2^k - 1) \cdot \left(2^{(m-1)k} + \cdots + 2^{3k} + 2^{2k} + 2^k + 1\right).$$

However, $2^{11} - 1 = 2047 = 23 \times 89$.

Great Internet Mersenne Prime Search

Testing primality of $2^p - 1$ is (relatively) easy: Lucas–Lehmer test.

GIMPS is a distributed computing project begun in 1996.
`http://mersenne.org`

All 50 known Mersenne primes:

$2^{2}-1$, $2^{3}-1$, $2^{5}-1$, $2^{7}-1$, $2^{13}-1$, $2^{17}-1$, $2^{19}-1$, $2^{31}-1$, $2^{61}-1$, $2^{89}-1$, $2^{107}-1$, $2^{127}-1$, $2^{521}-1$, $2^{607}-1$, $2^{1279}-1$, $2^{2203}-1$, $2^{2281}-1$, $2^{3217}-1$, $2^{4253}-1$, $2^{4423}-1$, $2^{9689}-1$, $2^{9941}-1$, $2^{11213}-1$, $2^{19937}-1$, $2^{21701}-1$, $2^{23209}-1$, $2^{44497}-1$, $2^{86243}-1$, $2^{110503}-1$, $2^{132049}-1$, $2^{216091}-1$, $2^{756839}-1$, $2^{859433}-1$, $2^{1257787}-1$, $2^{1398269}-1$, $2^{2976221}-1$, $2^{3021377}-1$, $2^{6972593}-1$, $2^{13466917}-1$, $2^{20996011}-1$, $2^{24036583}-1$, $2^{25964951}-1$, $2^{30402457}-1$, $2^{32582657}-1$, $2^{37156667}-1$, $2^{42643801}-1$, $2^{43112609}-1$, $2^{57885161}-1$, $2^{74207281}-1$, $2^{77232917}-1$

Largest known prime: $2^{77232917}-1$.
It was discovered in December 2017 and has 23,249,425 digits.

A prime-generating double exponential

In 1947, William Mills proved the existence of a real number α such that $\lfloor \alpha^{3^n} \rfloor$ is prime for $n \geq 1$.

If the Riemann hypothesis is true, the smallest such α is

$$\alpha = 1.306377883863080690468614492602605 7 \cdots$$

and generates the primes

$$2, 11, 1361, 2521008887, 1602223620400981813183132018 3, \ldots.$$

But the only known way of computing digits of α is by working backward from known large primes!

Euler's polynomial (1772)

Euler observed that $n^2 - n + 41$ is prime for $1 \leq n \leq 40$:

$$41, 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251,$$
$$281, 313, 347, 383, 421, 461, 503, 547, 593, 641, 691, 743, 797, 853,$$
$$911, 971, 1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601$$

But for $n = 41$ the value is $1681 = 41^2$.

Does there exist a polynomial $f(n)$ that only takes on prime values?

Yes. The constant polynomial $f(n) = 3$ does!

Prime-generating polynomials

What about a non-constant polynomial?

Suppose $f(n)$ is prime for all $n \geq 1$.
Let $p = f(1)$. Then

$$f(1 + px) = f(1) + p \times \text{(higher order terms)}$$

is divisible by p for each $x \geq 1$.

No.

What about a multivariate polynomial?

Theorem (Jones–Sato–Wada–Wiens 1976)

The set of positive values taken by the following degree-25 polynomial in 26 variables is equal to the set of prime numbers.

$$
\begin{aligned}
(k+2) \times (1 &- (wz+h+j-q)^2 \\
&- ((gk+2g+k+1)(h+j)+h-z)^2 \\
&- (2n+p+q+z-e)^2 \\
&- (16(k+1)^3(k+2)(n+1)^2+1-f^2)^2 \\
&- (e^3(e+2)(a+1)^2+1-o^2)^2 \\
&- ((a^2-1)y^2+1-x^2)^2 \\
&- (16r^2y^4(a^2-1)+1-u^2)^2 \\
&- (((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2)^2 \\
&- (n+l+v-y)^2 \\
&- ((a^2-1)l^2+1-m^2)^2 \\
&- (ai+k+1-l-i)^2 \\
&- (p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m)^2 \\
&- (q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x)^2 \\
&- (z+pl(a-p)+t(2ap-p^2-1)-pm)^2)
\end{aligned}
$$

Corollary: If $k+2$ is prime, then there is a proof that $k+2$ is prime consisting of 87 additions and multiplications.

Programming with polynomials

The set of positive values taken by

$$n \cdot \left(1 - (n - 2m)^2\right)$$

for positive integers n, m is the set of positive even numbers:

$$n - 2m = 0 \text{ has a solution } m \quad \Leftrightarrow \quad n \text{ is even.}$$

Given a system of equations whose solutions characterize primes, we can use the same trick.

The JSWW multivariate polynomial is an implementation of a primality test in the "programming language" of polynomials.

Hilbert's 10th problem

Hilbert's 10th problem

Is there an algorithm to determine whether a polynomial equation has positive integer solutions?

$$
\begin{aligned}
x^2 = y^2 + 2 &\quad\to\quad \text{no solution} \\
x^2 = y^2 + 3 &\quad\to\quad \text{solution exists } (x = 2, y = 1) \\
x^3 + y^3 = z^3 &\quad\to\quad \text{no solution} \\
x^3 + xy + 1 = y^4 &\quad\to\quad ???
\end{aligned}
$$

Work of Davis, Matiyasevich, Putnam, and Robinson, 1950–1970:
No. Any set of positive integers output by a computer program (running forever) can be encoded as the set of positive values of a polynomial.

And the halting problem is undecidable (Turing).

But where are the primes?

In practice, those “formulas for primes” don’t generate primes at all!

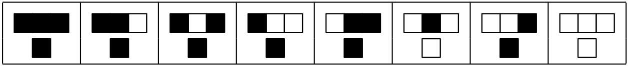
They are engineered to generate primes we already knew.

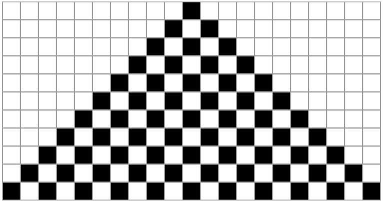
Are there formulas that generate primes we didn’t already know?

— INTERLUDE —

Cellular automata

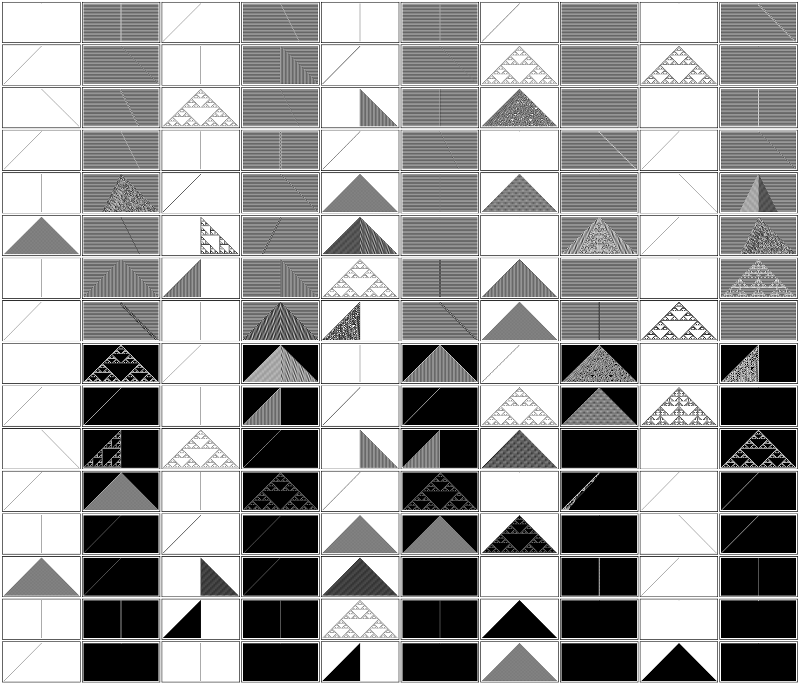
- alphabet Σ; for example $\Sigma = \{\square, \blacksquare\}$
- infinite row of elements of Σ (the initial condition)
- function $f : \Sigma^d \to \Sigma$ (the local update rule)

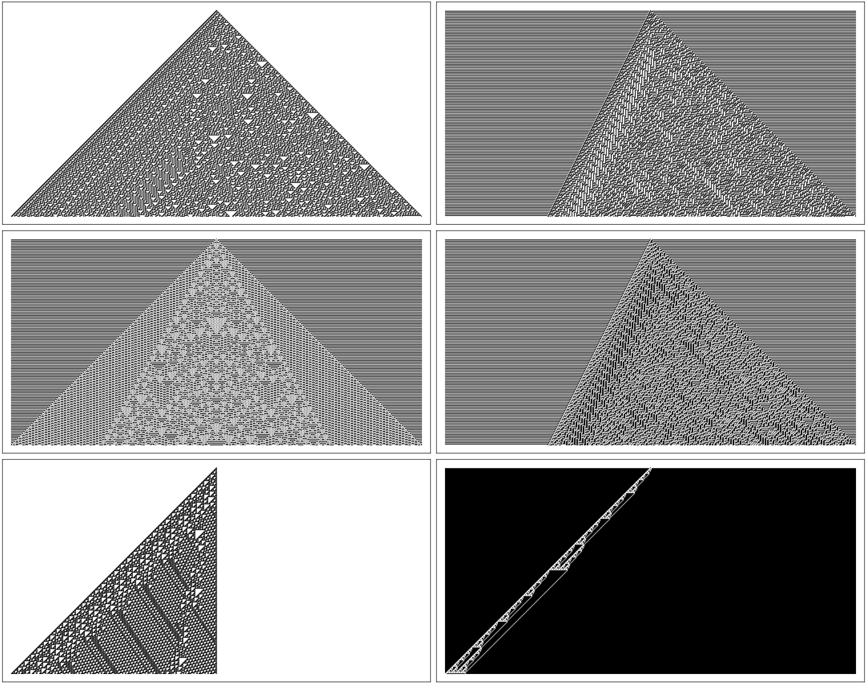




Conway's "game of life" is a famous 2-dimensional cellular automaton.

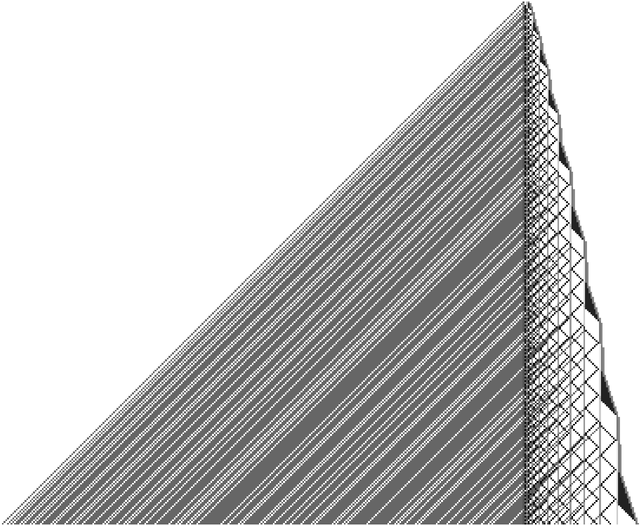
Stephen Wolfram's approach: Look at all possible rules.





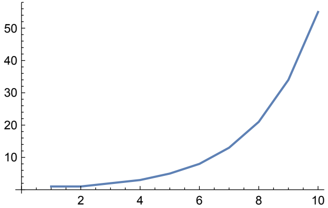
Programming with cellular automata

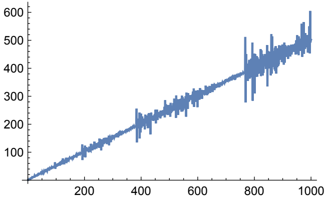
A 16-color rule depending on 3 cells that computes the primes:



At a summer research program in 2003, Wolfram conducted a live experiment on the space of all possible recurrences.

Fibonacci recurrence:
$s(n) = s(n-1) + s(n-2)$



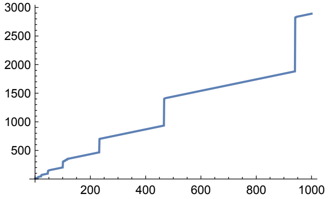


Hofstadter recurrence:
$s(n) = s(n - s(n-1)) + s(n - s(n-2))$

A new recurrence

Matthew Frank substituted other components...

One picture caught his eye:
$s(n) = s(n-1) + \gcd(n, s(n-1))$



$$\begin{aligned}
s(1) &= 7 \\
s(2) &= 7 + \gcd(2,7) = 7 + 1 = 8 \\
s(3) &= 8 + \gcd(3,8) = 8 + 1 = 9 \\
s(4) &= 9 + \gcd(4,9) = 9 + 1 = 10 \\
s(5) &= 10 + \gcd(5,10) = 10 + 5 = 15 \\
s(6) &= 15 + \gcd(6,15) = 15 + 3 = 18 \\
s(7) &= 18 + \gcd(7,18) = 18 + 1 = 19 \\
s(8) &= 19 + \gcd(8,19) = 19 + 1 = 20 \\
s(9) &= 20 + \gcd(9,20) = 20 + 1 = 21 \\
s(10) &= 21 + \gcd(10,21) = 21 + 1 = 22 \\
s(11) &= 22 + \gcd(11,22) = 22 + 11 = 33 \\
s(12) &= 33 + \gcd(12,33) = 33 + 3 = 36 \\
s(13) &= 36 + \gcd(13,36) = 36 + 1 = 37
\end{aligned}$$

Difference sequence $s(n) - s(n-1) = \gcd(n, s(n-1))$:

1, 1, 1, 5, 3, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 23, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 47, 3, 1, 5, ...

The difference sequence

1, 1, 1, 5, 3, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 23, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 47, 3, 1, 5, 3,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 101,
3, 1, 1, 7, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 13, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
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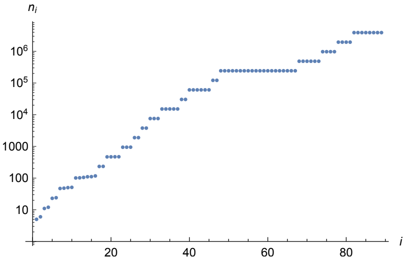
$s(n) - s(n-1)$ appears to always be 1 or prime!

Key observations



log plot of the position n_i of the ith non-1 term

- Ratio between clusters is very nearly 2.
- Each cluster is initiated by a large prime p.

If $\gcd(n_i, s(n_i - 1))$ is prime, then the next prime p that occurs is the smallest prime divisor of $2n_i - 1$, and it occurs at $n_{i+1} = n_i + \frac{p-1}{2}$.

Prime-generating recurrence

Theorem (Rowland 2008)

Let $s(1) = 7$*, and for* $n > 1$ *let*

$$s(n) = s(n-1) + \gcd(n, s(n-1)).$$

For each $n \geq 2$*,* $\gcd(n, s(n-1))$ *is either* 1 *or prime.*

This recurrence can generate primes we didn't expect to see!

Does it generate primes efficiently? No.
Each prime p is preceded by $\frac{p-3}{2}$ consecutive 1s.

Which primes appear?

5, 3, 11, 3, 23, 3, 47, 3, 5, 3, 101, 3, 7, 11, 3, 13, 233, 3, 467, 3, 5, 3, 941, 3, 7, 1889, 3, 3779, 3, 7559, 3, 13, 15131, 3, 53, 3, 7, 30323, 3, 60647, 3, 5, 3, 101, 3, 121403, 3, 242807, 3, 5, 3, 19, 7, 5, 3, 47, 3, 37, 5, 3, 17, 3, 199, 53, 3, 29, 3, 486041, 3, 7, 421, 23, 3, 972533, 3, 577, 7, 1945649, 3, 163, 7, 3891467, 3, 5, 3, 127, 443, 3, 31, 7783541, 3, 7, 15567089, 3, 19, 29, 3, 5323, 7, 5, 3, 31139561, 3, 41, 3, 5, 3, 62279171, 3, 7, 83, 3, 19, 29, 3, 1103, 3, 5, 3, 13, 7, 124559609, 3, 107, 3, 911, 3, 249120239, 3, 11, 3, 7, 61, 37, 179, 3, 31, 19051, 7, 3793, 23, 3, 5, 3, 6257, 3, 43, 11, 3, 13, 5, 3, 739, 37, 5, 3, 498270791, 3, 19, 11, 3, 41, 3, 5, 3, 996541661, 3, 7, 37, 5, 3, 67, 1993083437, 3, 5, 3, 83, 3, 5, 3, 73, 157, 7, 5, 3, 13, 3986167223, 3, 7, 73, 5, 3, 7, 37, 7, 11, 3, 13, 17, 3, 19, 29, 3, 13, 23, 3, 5, 3, 11, 3, 7972334723, 3, 7, 463, 5, 3, 31, 7, 3797, 3, 5, 3, 15944673761, 3, 11, 3, 5, 3, 17, 3, 53, 3, 139, 607, 17, 3, 5, 3, 11, 3, 7, 113, 3, 11, 3, 5, 3, 293, 3, 5, 3, 53, 3, 5, 3, 151, 11, 3, 31889349053, 3, 63778698107, 3, 5, 3, 491, 3, 1063, 5, 3, 11, 3, 7, 13, 29, 3, 6899, 3, 13, 127557404753, 3, 41, 3, 373, 19, 11, 3, 43, 17, 3, 320839, 255115130849, 3, 510230261699, 3, 72047, 3, 53, 3, 17, 3, 67, 5, 3, 79, 157, 5, 3, 110069, 3, 7, 1020460705907, 3, 5, 3, 43, 179, 3, 557, 3, 167, ...

$p = 2$ cannot occur. But all odd primes below 587 do occur.

Theorem (Chamizo–Raboso–Ruiz-Cabello 2011)

The difference sequence contains infinitely many distinct primes.

A variant

Benoit Cloitre looked at the recurrence

$$s(n) = s(n-1) + \mathrm{lcm}(n, s(n-1))$$

with $s(1) = 1$.

He observed that $\frac{s(n)}{s(n-1)} - 1$ seems to be 1 or prime for each $n \geq 2$:

2, 1, 2, 5, 1, 7, 1, 1, 5, 11, 1, 13, 1, 5, 1, 17, 1, 19, 1, 1, 11, 23, 1, 5, 13, 1, 1, 29, 1, 31, 1, 11, 17, 1, 1, 37, 1, 13, 1, 41, 1, 43, 1, 1, 23, 47, 1, 1, 1, 17, 13, 53, 1, 1, 1, 1, 29, 59, 1, 61, 1, 1, 1, 13, 1, 67, 1, 23, 1, 71, 1, 73, 1, 1, 1, 1, 13, 79, 1, 1, 41, 83, 1, 1, 43, 29, 1, 89, 1, 13, 23, 1, 47, 1, 1, 97, 1, 1, 1, 101, 1, 103, 1, 1, 53, 107, 1, 109, 1, 1, 1, 113, 1, 23, 29, 1, 59, 1, 1, 1, 61, 41, 1, 1, 1, 127, 1, 43, 1, 131, 1, 1, 67, 1, 1, 137, 1, 139, 1, 47, 71, 1, 1, 29, 73, 1, 1, 149, 1, 151, 1, 1, 1, 1, 1, 157, 1, 53, 1, 1, 1, 163, 1, 1, 83, ...

No proof yet!