Formulas for Primes

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The sequence of primes

$\cancel{1}\ 2\ 3\ \cancel{4}\ 5\ \cancel{6}\ 7\ \cancel{8}\ \cancel{9}\ \cancel{10}\ 11\ \cancel{12}\ 13\ \cancel{14}\ \cancel{15}\ \cancel{16}\ 17\ \cancel{18}\ 19\ \cancel{20}\ \cancel{21}\ \cancel{22}\ 23\ \cancel{24}\ \cancel{25} \cdots$

The sieve of Eratosthenes generates the sequence of primes:

$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, \ldots$

Two questions:

- Is it easy to tell when a number is prime?
- Is it easy to generate primes? Are there formulas that always produce primes?

Can primality be determined quickly?

Trial division: Test divisibility by all numbers $2 \leq m \leq \sqrt{n}$.

Wilson's Theorem (Lagrange, 1773)

If $n \geq 2$, then n is prime if and only if n divides $(n-1)! + 1$.

For example, 5 divides $4! + 1 = 25$, but 6 doesn't divide $5! + 1 = 121$.

But determining whether an ℓ-digit number is prime using Wilson's theorem requires multiplying $\approx 10^\ell$ numbers.

Is there a polynomial-time algorithm for testing primality? Yes.

Agrawal, Kayal, & Saxena (2002) provided an algorithm that determines whether an ℓ-digit number is prime in $c \cdot \ell^{12}$ steps.

In 1964, C. P. Willans produced this formula for the nth prime:

$$p_n = 1 + \sum_{i=1}^{2^n} \left\lfloor \left(\frac{n}{\sum_{j=1}^{i} \left\lfloor \left(\cos \frac{(j-1)!+1}{j} \pi \right)^2 \right\rfloor} \right)^{1/n} \right\rfloor$$

Willans' formula is built on Wilson's theorem!

$$\frac{(j-1)!+1}{j} = \begin{cases} \text{an integer} & \text{if } j = 1 \text{ or } j \text{ is prime} \\ \text{not an integer} & \text{if } j \geq 2 \text{ is not prime.} \end{cases}$$

$$\left\lfloor \left(\cos \frac{(j-1)!+1}{j} \pi \right)^2 \right\rfloor = \begin{cases} 1 & \text{if } j = 1 \text{ or } j \text{ is prime} \\ 0 & \text{if } j \geq 2 \text{ is not prime.} \end{cases}$$

$$\sum_{j=1}^{i} \left\lfloor \left(\cos \frac{(j-1)!+1}{j} \pi \right)^2 \right\rfloor = \pi(i) + 1.$$

$$\left\lfloor \left(\frac{n}{\pi(i)+1} \right)^{1/n} \right\rfloor = \begin{cases} 1 & \text{if } i < p_n \\ 0 & \text{if } i \geq p_n. \end{cases}$$

An exponential formula

n	0	1	2	3	4	5	6	7	8	9	10
$2^n + 1$	2	3	5	9	17	33	65	129	257	513	1025

If $2^n + 1$ is prime and $n \geq 1$, must n be a power of 2?

$$a^m - b^m = (a - b) \cdot \left(a^{m-1} + a^{m-2}b + a^{m-3}b^2 + \cdots + b^{m-1}\right)$$

Suppose n is divisible by some odd number m.
Then letting $a = 2^{n/m}$ and $b = -1$ shows that
$a - b = 2^{n/m} + 1$ divides $a^m - b^m = (2^{n/m})^m - (-1)^m = 2^n + 1$.

Yes.

And in fact $2^{16} + 1 = 65537$ is prime!

Fermat primes

$$2^{2^0} + 1 = 3$$
$$2^{2^1} + 1 = 5$$
$$2^{2^2} + 1 = 17$$
$$2^{2^3} + 1 = 257$$
$$2^{2^4} + 1 = 65537$$

Fermat conjectured that $2^{2^n} + 1$ is prime for all $n \geq 0$.

However, Euler found that

$$2^{2^5} + 1 = 4294967297 = 641 \times 6700417.$$

And no Fermat primes have been found since!

Mersenne primes

n	0	1	2	3	4	5	6	7	8	9	10
$2^n - 1$	0	1	3	7	15	31	63	127	255	511	1023

Marin Mersenne (1588–1648)

If $2^n - 1$ is prime, must n be prime? Yes:

$$2^{km} - 1 = (2^k - 1) \cdot \left(2^{(m-1)k} + \cdots + 2^{3k} + 2^{2k} + 2^k + 1\right).$$

However, $2^{11} - 1 = 2047 = 23 \times 89$.

The Great Internet Mersenne Prime Search

Testing primality of $2^p - 1$ is (relatively) easy: Lucas–Lehmer test.

GIMPS is a distributed computing project begun in 1996.
`http://mersenne.org`

All 47 known Mersenne primes:

2^2-1, 2^3-1, 2^5-1, 2^7-1, $2^{13}-1$, $2^{17}-1$, $2^{19}-1$, $2^{31}-1$, $2^{61}-1$, $2^{89}-1$, $2^{107}-1$, $2^{127}-1$, $2^{521}-1$, $2^{607}-1$, $2^{1279}-1$, $2^{2203}-1$, $2^{2281}-1$, $2^{3217}-1$, $2^{4253}-1$, $2^{4423}-1$, $2^{9689}-1$, $2^{9941}-1$, $2^{11213}-1$, $2^{19937}-1$, $2^{21701}-1$, $2^{23209}-1$, $2^{44497}-1$, $2^{86243}-1$, $2^{110503}-1$, $2^{132049}-1$, $2^{216091}-1$, $2^{756839}-1$, $2^{859433}-1$, $2^{1257787}-1$, $2^{1398269}-1$, $2^{2976221}-1$, $2^{3021377}-1$, $2^{6972593}-1$, $2^{13466917}-1$, $2^{20996011}-1$, $2^{24036583}-1$, $2^{25964951}-1$, $2^{30402457}-1$, $2^{32582657}-1$, $2^{37156667}-1$, $2^{42643801}-1$, $2^{43112609}-1$

Largest known prime: $2^{43112609} - 1$.
It was discovered in August 2008 and has 12978189 decimal digits.

A prime-generating double exponential

In 1947, William Mills proved the existence of a real number b such that $\lfloor b^{3^n} \rfloor$ is prime for $n \geq 1$.

If the Riemann hypothesis is true, the smallest such b is

$$b = 1.3063778838630806904686144926026057 \cdots$$

and generates the primes

$$2, 11, 1361, 2521008887, 16022236204009818131831320183, \ldots.$$

But the only known way of computing digits of b is by working backward from known large primes!

Euler's polynomial (1772)

Euler observed that $n^2 - n + 41$ is prime for $1 \leq n \leq 40$:

$$41, 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251,$$
$$281, 313, 347, 383, 421, 461, 503, 547, 593, 641, 691, 743, 797, 853,$$
$$911, 971, 1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601$$

But for $n = 41$ the value is $1681 = 41^2$.

Does there exist a polynomial $f(n)$ that only takes on prime values?

Yes. The constant polynomial $f(n) = 3$ does!

Prime-generating polynomials

What about a non-constant polynomial?

Suppose $f(n)$ is prime for all $n \geq 1$.
Let $p = f(1)$.
Then $f(1 + pk) = f(1) + p \times$ (higher order terms),
so p divides $f(1 + pk)$ for each $k \geq 1$.

No.

What about a multivariate polynomial?

Theorem (Jones–Sato–Wada–Wiens, 1976)

The set of positive values taken by the following degree-25 polynomial in 26 variables is equal to the set of prime numbers.

$$
\begin{aligned}
(k+2)(1 &- (wz+h+j-q)^2 \\
&- ((gk+2g+k+1)(h+j)+h-z)^2 \\
&- (2n+p+q+z-e)^2 \\
&- (16(k+1)^3(k+2)(n+1)^2+1-f^2)^2 \\
&- (e^3(e+2)(a+1)^2+1-o^2)^2 \\
&- ((a^2-1)y^2+1-x^2)^2 \\
&- (16r^2y^4(a^2-1)+1-u^2)^2 \\
&- (((a+u^2(u^2-a))^2-1)(n+4dy)^2+1-(x+cu)^2)^2 \\
&- (n+l+v-y)^2 \\
&- ((a^2-1)l^2+1-m^2)^2 \\
&- (ai+k+1-l-i)^2 \\
&- (p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m)^2 \\
&- (q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x)^2 \\
&- (z+pl(a-p)+t(2ap-p^2-1)-pm)^2)
\end{aligned}
$$

Corollary: If $k+2$ is prime, then there is a proof that $k+2$ is prime consisting of 87 additions and multiplications.

Programming with polynomials

The set of positive values taken by

$$n \cdot \left(1 - (n - 2m)^2\right)$$

as n and m run over the positive integers is the set of positive even numbers:

$$n \text{ is even} \quad \Leftrightarrow \quad n = 2m \text{ has a solution in integers.}$$

Given a system of equations whose integer solutions characterize primes, you can use the same trick.

The multivariate polynomial is an implementation of a primality test in the "programming language" of polynomials.

Hilbert's 10th problem

Hilbert's 10th problem

Is there an algorithm to determine whether a polynomial equation has positive integer solutions?

$$
\begin{aligned}
x^2 = y^2 + 2 \quad &\to \quad \text{no solution} \\
x^2 = y^2 + 3 \quad &\to \quad \text{solution exists} \\
x^3 + y^3 = z^3 \quad &\to \quad \text{no solution} \\
x^3 + xy + 1 = y^4 \quad &\to \quad ???
\end{aligned}
$$

Work of Davis, Matiyasevich, Putnam, and Robinson, 1950–1970:
No. Any set of positive integers output by a computer program (running forever) can be encoded as the set of positive values of a polynomial.

But where are the primes?

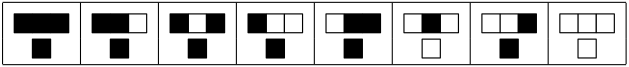
In practice, none of those “generators” actually generate primes at all!

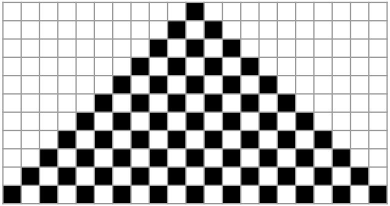
They are *engineered* to generate primes we already knew.

Are there formulas that generate primes we didn’t already know?

Cellular automata

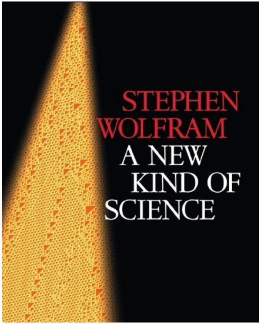
- alphabet Σ (for example, $\Sigma = \{\square, \blacksquare\}$)
- function $i : \mathbb{Z} \to \Sigma$ (the initial condition)
- function $f : \Sigma^d \to \Sigma$ (the local update rule)



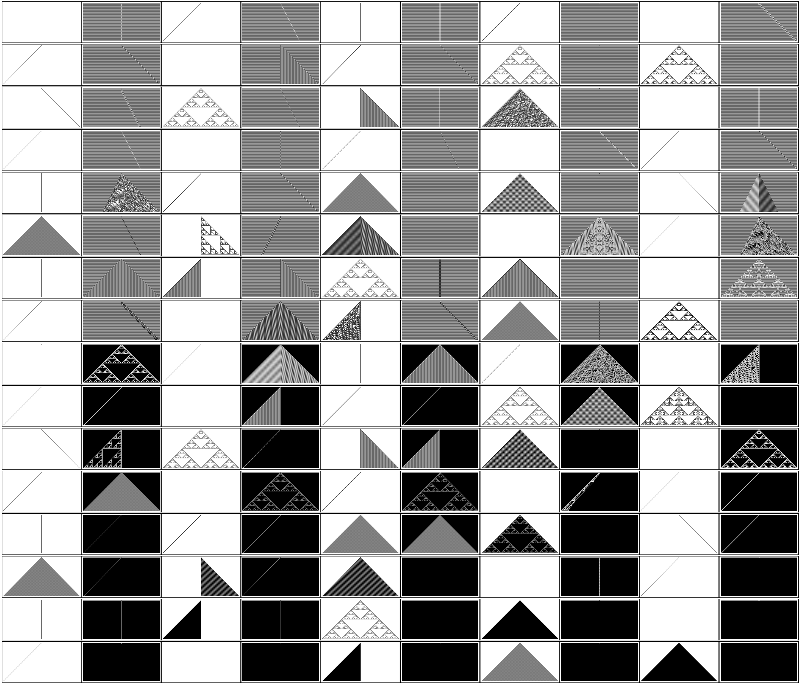


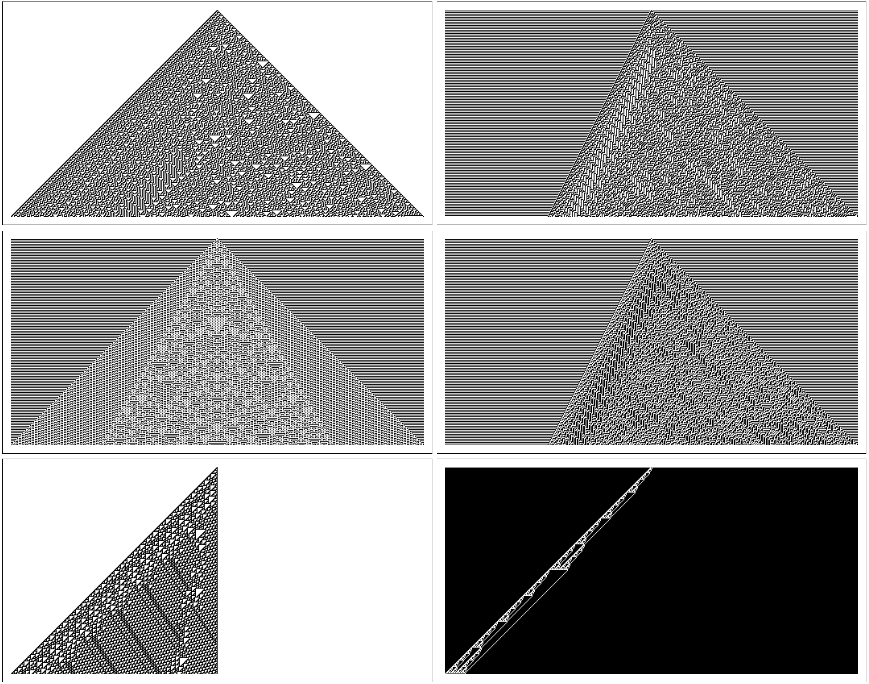
The “game of life” is a famous two-dimensional cellular automaton.

In 2002 Stephen Wolfram published a book on simple programs, including cellular automata.



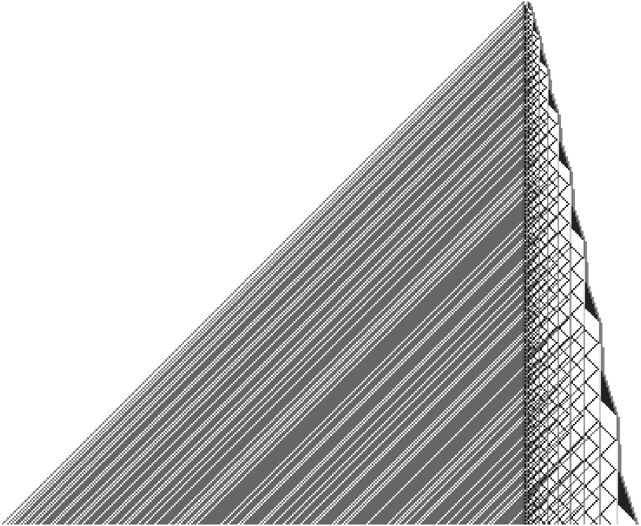
Wolfram's approach: Systematically look at all possible rules.





A prime-generating cellular automaton

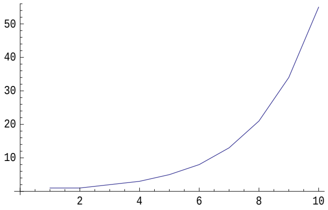
A 16-color rule depending on 3 cells that computes the primes:

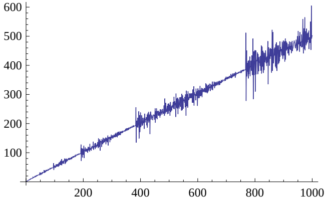


Recurrences

In 2003, Matthew Frank applied the methodology to recurrences.

Fibonacci recurrence:
$s(n) = s(n-1) + s(n-2)$



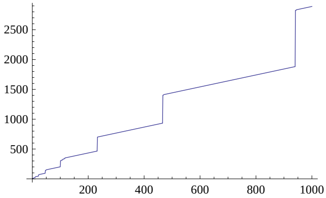


Hofstadter recurrence:
$s(n) = s(n - s(n-1)) + s(n - s(n-2))$

A new recurrence

Frank systematically substituted different functions into a template recurrence and looked at the pictures they generated.

This one caught his eye:



$$s(n) = s(n-1) + \gcd(n, s(n-1))$$

First few terms

$$s(n) = s(n-1) + \gcd(n, s(n-1))$$

$s(1) = 7$
$s(2) = 7 + \gcd(2, 7) = 7 + 1 = 8$
$s(3) = 8 + \gcd(3, 8) = 8 + 1 = 9$
$s(4) = 9 + \gcd(4, 9) = 9 + 1 = 10$
$s(5) = 10 + \gcd(5, 10) = 10 + 5 = 15$
$s(6) = 15 + \gcd(6, 15) = 15 + 3 = 18$
$s(7) = 18 + \gcd(7, 18) = 18 + 1 = 19$
$s(8) = 19 + \gcd(8, 19) = 19 + 1 = 20$
$s(9) = 20 + \gcd(9, 20) = 20 + 1 = 21$
$s(10) = 21 + \gcd(10, 21) = 21 + 1 = 22$
$s(11) = 22 + \gcd(11, 22) = 22 + 11 = 33$
$s(12) = 33 + \gcd(12, 33) = 33 + 3 = 36$
$s(13) = 36 + \gcd(13, 36) = 36 + 1 = 37$
$s(14) = 37 + \gcd(14, 37) = 37 + 1 = 38$

Difference sequence $s(n) - s(n-1) = \gcd(n, s(n-1))$:

1, 1, 1, 5, 3, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 23, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 47, 3, 1, 5, ...

The sequence $\gcd(n, s(n-1))$

1, 1, 1, 5, 3, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 23, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 47, 3, 1, 5, 3,
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 101,
3, 1, 1, 7, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 13, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
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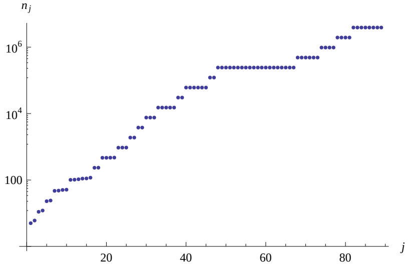
$\gcd(n, s(n-1))$ appears to always be 1 or prime!

Key observations



logarithmic plot of n_j,
the jth value of n for which
$\gcd(n, s(n-1)) \neq 1$

- Ratio between clusters is very nearly 2.
- Each cluster is initiated by a large prime p.

In fact, if $\gcd(n_j, s(n_j - 1))$ is prime, then the next prime that occurs is the smallest prime divisor p of $2n_j - 1$, and it occurs at $n_{j+1} = n_j + \frac{p-1}{2}$.

Prime-generating recurrence

Theorem (Rowland, 2008)

Let $s(1) = 7$, and for $n > 1$ let

$$s(n) = s(n-1) + \gcd(n, s(n-1)).$$

For each $n \geq 2$, $\gcd(n, s(n-1))$ is either 1 or prime.

This recurrence can generate primes we didn't expect to see!

Does it generate primes efficiently? No.

Without the shortcut, $\frac{p-3}{2}$ consecutive 1s precede p.
With the shortcut, each step requires finding the smallest prime divisor of $2n-1$.

Other initial conditions

Do all initial conditions produce only 1s and primes? No.
$s(1) = 532$ produces $\gcd(18, s(17)) = \gcd(18, 567) = 9$.
$s(1) = 801$ produces $\gcd(21, s(20)) = \gcd(21, 840) = 21$.

Conjecture

Let $n_1 \geq 1$ and $s(n_1) \geq 1$. For $n > n_1$ let

$$s(n) = s(n-1) + \gcd(n, s(n-1)).$$

Then there exists an N such that, for each $n > N$, $\gcd(n, s(n-1))$ is either 1 or prime.

It would suffice to show that $s(n)/n$ always reaches 1, 2, or 3.

Which primes appear?

5, 3, 11, 3, 23, 3, 47, 3, 5, 3, 101, 3, 7, 11, 3, 13, 233, 3, 467, 3, 5, 3, 941, 3, 7, 1889, 3, 3779, 3, 7559, 3, 13, 15131, 3, 53, 3, 7, 30323, 3, 60647, 3, 5, 3, 101, 3, 121403, 3, 242807, 3, 5, 3, 19, 7, 5, 3, 47, 3, 37, 5, 3, 17, 3, 199, 53, 3, 29, 3, 486041, 3, 7, 421, 23, 3, 972533, 3, 577, 7, 1945649, 3, 163, 7, 3891467, 3, 5, 3, 127, 443, 3, 31, 7783541, 3, 7, 15567089, 3, 19, 29, 3, 5323, 7, 5, 3, 31139561, 3, 41, 3, 5, 3, 62279171, 3, 7, 83, 3, 19, 29, 3, 1103, 3, 5, 3, 13, 7, 124559609, 3, 107, 3, 911, 3, 249120239, 3, 11, 3, 7, 61, 37, 179, 3, 31, 19051, 7, 3793, 23, 3, 5, 3, 6257, 3, 43, 11, 3, 13, 5, 3, 739, 37, 5, 3, 498270791, 3, 19, 11, 3, 41, 3, 5, 3, 996541661, 3, 7, 37, 5, 3, 67, 1993083437, 3, 5, 3, 83, 3, 5, 3, 73, 157, 7, 5, 3, 13, 3986167223, 3, 7, 73, 5, 3, 7, 37, 7, 11, 3, 13, 17, 3, 19, 29, 3, 13, 23, 3, 5, 3, 11, 3, 7972334723, 3, 7, 463, 5, 3, 31, 7, 3797, 3, 5, 3, 15944673761, 3, 11, 3, 5, 3, 17, 3, 53, 3, 139, 607, 17, 3, 5, 3, 11, 3, 7, 113, 3, 11, 3, 5, 3, 293, 3, 5, 3, 53, 3, 5, 3, 151, 11, 3, 31889349053, 3, 63778698107, 3, 5, 3, 491, 3, 1063, 5, 3, 11, 3, 7, 13, 29, 3, 6899, 3, 13, 127557404753, 3, 41, 3, 373, 19, 11, 3, 43, 17, 3, 320839, 255115130849, 3, 510230261699, 3, 72047, 3, 53, 3, 17, 3, 67, 5, 3, 79, 157, 5, 3, 110069, 3, 7, 1020460705907, 3, 5, 3, 43, 179, 3, 557, 3, 167, . . .

Which primes appear?

$p = 2$ cannot occur.
After ten thousand nontrivial gcds, the smallest odd prime that has not yet appeared is 587.

Theorem (Chamizo–Raboso–Ruiz-Cabello, 2011)

If $s(n) = s(n-1) + \gcd(n, s(n-1))$ *with* $s(1) = 7$*, then the difference sequence* $\gcd(n, s(n-1))$ *contains infinitely many distinct primes.*

Moreover, they obtained a simple characterization of the finite sequences of primes that occur for some initial condition.
For example, the sequence $17, 5, p$ does not occur for any prime $p > 3$.

It also follows that no sequence of primes occurs twice consecutively.

A variant

Benoit Cloitre looked at the recurrence

$$s(n) = s(n-1) + \operatorname{lcm}(n, s(n-1))$$

with $s(1) = 1$.

He observed that $\frac{s(n)}{s(n-1)} - 1$ seems to be 1 or prime for each $n \geq 2$:

2, 1, 2, 5, 1, 1, 1, 1, 5, 11, 1, 13, 1, 5, 1, 17, 1, 19, 1, 1, 11, 23, 1, 5, 13, 1, 1, 29, 1, 31, 1, 11, 17, 1, 1, 37, 1, 13, 1, 41, 1, 43, 1, 1, 23, 47, 1, 1, 1, 17, 13, 53, 1, 1, 1, 1, 29, 59, 1, 61, 1, 1, 1, 13, 1, 67, 1, 23, 1, 71, 1, 73, 1, 1, 1, 1, 13, 79, 1, 1, 41, 83, 1, 1, 43, 29, 1, 89, 1, 13, 23, 1, 47, 1, 1, 97, 1, 1, 1, 101, 1, 103, 1, 1, 53, 107, 1, 109, 1, 1, 1, 113, 1, 23, 29, 1, 59, 1, 1, 1, 61, 41, 1, 1, 1, 127, 1, 43, 1, 131, 1, 1, 67, 1, 1, 137, 1, 139, 1, 47, 71, 1, 1, 29, 73, 1, 1, 149, 1, 151, 1, 1, 1, 1, 1, 157, 1, 53, 1, 1, 1, 163, 1, 1, 83, ...

Conjecturally, every prime appears except 3 and 7.

No proof yet!