Extremal words avoiding partial repetitions

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Squares on a binary alphabet

Axel Thue (1863–1922)

A square is a nonempty word of the form *ww*.

Are squares are avoidable on a binary alphabet?
Are there arbitrarily long square-free words on $\{0, 1\}$?

$$010\boxtimes$$

Squares on a 3-letter alphabet

Are squares avoidable on $\{0, 1, 2\}$?

$$01020120210120102012021020102101201020120210\cdots$$

Theorem (Thue 1906)

There exist arbitrarily long square-free words on 3 *letters.*

The backtracking algorithm builds the lexicographically least word.

Open problem (Allouche–Shallit, *Automatic Sequences* §1.10)

Characterize the lex. least square-free word over $\{0, 1, 2\}$.

Overlaps

An overlap is a word of the form *cxcxc*, where *c* is a letter; equivalently, *wwc*, where *c* is the first letter of *w*.

Theorem (Thue 1912)

There exist arbitrarily long overlap-free words on 2 letters.

Let $\varphi(0) = 01$ and $\varphi(1) = 10$.
The Thue–Morse word is overlap-free:

$$\varphi^\infty(0) = 01101001100101101001011001101001\cdots$$

It follows that cubes are avoidable on a binary alphabet.

Thue–Morse word

n	0	1	2	3	4	5	6	7
$(n)_2$	ϵ	1	10	11	100	101	110	111
$t_n = \|(n)_2\|_\Sigma \bmod 2$	0	1	1	0	1	0	0	1

- Curious products (Robbins 1958):

$$\prod_{n\geq 0}(n+1)^{(-1)^{t_n}} = \frac{1\cdot 4\cdot 6\cdot 7\cdot 10\cdot 11\cdot 13\cdot 16\cdots}{2\cdot 3\cdot 5\cdot 8\cdot 9\cdot 12\cdot 14\cdot 15\cdots} = \frac{1}{\sqrt{2}}$$

- Multigrades (Prouhet 1851):

$$0^0+3^0+5^0+6^0 = 1^0+2^0+4^0+7^0 = 4$$
$$0^1+3^1+5^1+6^1 = 1^1+2^1+4^1+7^1 = 14$$
$$0^2+3^2+5^2+6^2 = 1^2+2^2+4^2+7^2 = 70$$

Theorem (Allouche–Currie–Shallit 1998)

The lex. least overlap-free word on $\{0,1\}$ *is* $001001\varphi^\infty(1)$.

Infinite alphabet

On an infinite alphabet, the backtracking algorithm doesn't backtrack.

Are squares avoidable on $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$? Yes.

$$010201030102010401020103010201050 \cdots$$

Theorem (Guay-Paquet–Shallit 2009)

Let $\varphi(n) = 0(n+1)$. The lex. least square-free word on $\mathbb{Z}_{\geq 0}$ is $\varphi^\infty(0)$.

The nth letter of each word is $\nu_2(n+1)$. The k-adic valuation $\nu_k(n)$ is the exponent of the highest power of k dividing n.

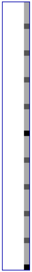
n	0	1	2	3	4	5	6	7
$(n+1)_2$	1	10	11	100	101	110	111	1000
$\nu_2(n+1)$	0	1	0	2	0	1	0	3

Integer powers

More generally, let $a \geq 2$. Let $\varphi(n) = 0^{a-1}(n+1)$.
The lex. least a-power-free word on $\mathbb{Z}_{\geq 0}$ is $\varphi^\infty(0) = (\nu_a(n+1))_{n \geq 0}$.

$$\mathbf{w}_5 = 00001000010000100001000020000100001 \cdots$$

$$\begin{array}{rl} \mathbf{w}_5 = & 00001 \\ & 00001 \\ & 00001 \\ & 00001 \\ & 00002 \\ & 00001 \\ & \vdots \end{array}$$



$$\begin{aligned} w(5n+0) &= 0 \\ w(5n+1) &= 0 \\ w(5n+2) &= 0 \\ w(5n+3) &= 0 \\ w(5n+4) &= w(n)+1 \end{aligned}$$

The letters of $\mathbf{w}_5$ satisfy a recurrence with constant coefficients.

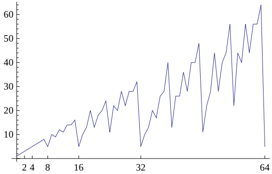
k-regular sequences

An integer sequence $s(n)_{n\geq 0}$ is *k-regular* if the set

$$\{s(k^e n + r) : e \geq 0 \text{ and } 0 \leq r \leq k^e - 1\}$$

is contained in a finite-dimensional $\mathbb{Q}$-vector space.

For example, let $s(n) = |\{0 \leq m \leq n : \binom{n}{m} \not\equiv 0 \mod 8\}|$.



$$\begin{aligned}
s(2n+1) &= 2s(n) \\
s(4n+0) &= s(2n) \\
s(8n+2) &= -2s(n) + 2s(2n) + s(4n+2) \\
s(8n+6) &= 2s(4n+2)
\end{aligned}$$

Uniqueness of k

A 2-regular sequence is also 4-regular, and vice versa.

k and l are multiplicatively dependent if there exist positive integers s and t such that $k^s = l^t$.

For example, 4 and 8 (since $4^3 = 8^2$).

Theorem (Bell 2006)

If k and l are multiplicatively independent and $s(n)_{n \geq 0}$ is both k-regular and l-regular, then $\sum_{n \geq 0} s(n) x^n$ is the power series of a rational function whose poles are roots of unity.

Fractional powers

$011101 = (0111)^{3/2}$ is a $\frac{3}{2}$-power.
If $|x| = |y| = |z|$, then $xyzxyzx = (xyz)^{7/3}$ is a $\frac{7}{3}$-power.

Definition

A word w is an $\frac{a}{b}$-power if

$$w = v^e x$$

where $e \geq 0$ is an integer, x is a prefix of v, and $\frac{|w|}{|v|} = \frac{a}{b}$.

Notation

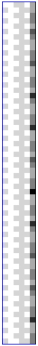
For $\frac{a}{b} > 1$, let $\mathbf{w}_{a/b}$ be the lex. least $\frac{a}{b}$-power-free word on $\mathbb{Z}_{\geq 0}$.

We assume $\gcd(a, b) = 1$ from now on.

Avoiding 3/2-powers

$$\mathbf{w}_{3/2} = 00110210011200110310011300110210011400110 3\cdots$$

$$\begin{aligned}\mathbf{w}_{3/2} = {} & 001102\\ & 100112\\ & 001103\\ & 100113\\ & 001102\\ & 100114\\ & 001103\\ & 100112\\ & \vdots\end{aligned}$$



Theorem (Rowland–Shallit 2012)

The sequence of letters in $\mathbf{w}_{3/2}$ *is* 6*-regular.*

Why 6?

The interval $\frac{a}{b} \geq 2$

$$\mathbf{w}_{5/2} = 00001000010000100001000020000100001 \cdots = \mathbf{w}_5$$

Theorem

If $\frac{a}{b} \geq 2$, then $\mathbf{w}_{a/b} = \mathbf{w}_a$.

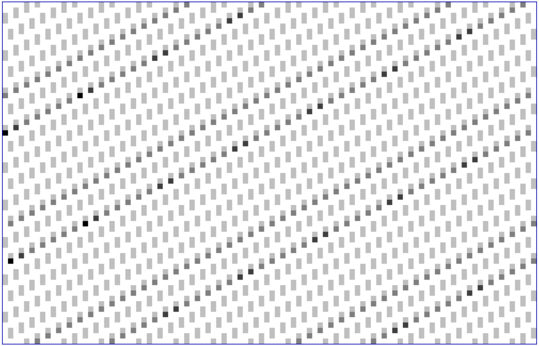
Proof (one direction).

The a-power $v^a = (v^b)^{a/b}$ is also an $\frac{a}{b}$-power.
So $\mathbf{w}_{a/b}$ is a-power-free. Thus $\mathbf{w}_a \leq \mathbf{w}_{a/b}$ lexicographically. $\square$

Therefore it suffices to consider $1 < \frac{a}{b} < 2$.

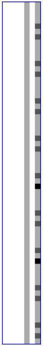
$\mathbf{w}_{5/3}$ wrapped into 100 columns

$$\mathbf{w}_{5/3} = 0000101000010100001010000101000010200000101\cdots$$



$\mathbf{w}_{5/3}$ wrapped into 7 columns

$$\mathbf{w}_{5/3} = 000010100001010000101000010100001020000101 \cdots$$

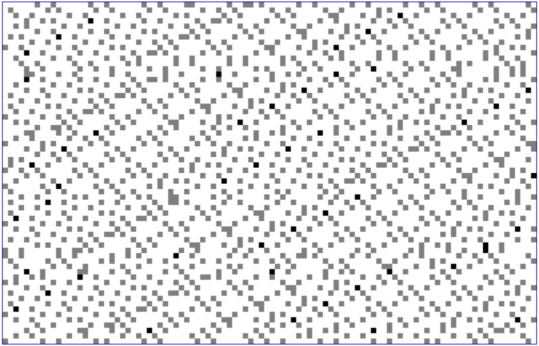


Theorem

$\mathbf{w}_{5/3} = \varphi^{\infty}(0)$, *where* $\varphi(n) = 000010(n+1)$ *is a* 7*-uniform morphism.*

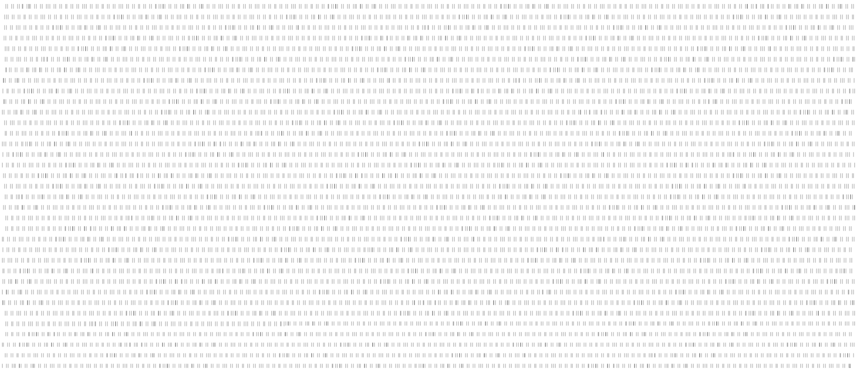
$\mathbf{w}_{7/4}$ wrapped into 100 columns

$$\mathbf{w}_{7/4} = 000000100100000001001000000010010000011000000\cdots$$



$\mathbf{w}_{7/4}$ wrapped into 50847 columns

$$\mathbf{w}_{7/4} = 0000001001000000100100000010010000011000000\cdots$$

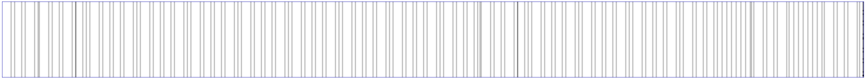


Theorem

$\mathbf{w}_{7/4} = \varphi^\infty(0)$ *for some* 50847*-uniform morphism* $\varphi(n) = u\,(n+2)$.

$\mathbf{w}_{8/5}$ wrapped into 733 columns

$$\mathbf{w}_{8/5} = 0000000100100000010010000000100110000000100\cdots$$



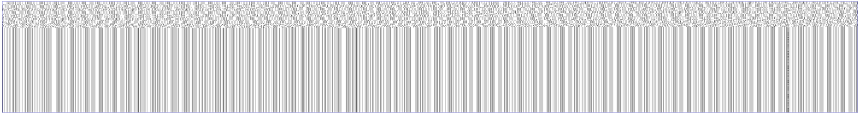
Theorem

$\mathbf{w}_{8/5} = \varphi^{\infty}(0)$ *for the* 733*-uniform morphism*

$$\begin{aligned}\varphi(n) = {} & 00000001001000000100100000000100110000000100100000010010000000010020000 \\ & 01001001000000001001000000100100000010010000000010010010000000010010000 0 \\ & 1001000000100100000000100100100000000100100000010010000001001000000001001 \\ & 00100000000100100000010010000001001000000001001001000000001001000000 10010 \\ & 0000100100000000100100100000000100100000010010000001001000000001001001 00 \\ & 0000010010000001001000000100100000000100100100000000100100000010010000 01 \\ & 0010110000000010010000001001000000001002000000100100100000000100100000 10 \\ & 0100000010010000000010010010000000010010000001001000000100100000000100100 \\ & 1000000010010000001001000000100100000000100100100000000100100000010010 00 \\ & 00100100010001000100010001000100011010000000010010000001001000000000101 \\ & 0001000100010001000100010001010000000010010000001001000000000101 00(n+2).\end{aligned}$$

$\mathbf{w}_{6/5}$ wrapped into 1001 columns

$$\mathbf{w}_{6/5} = 000001111102020201011101000202120210110010\cdots$$



There is a transient region.

Introduce a new letter $0'$, and let $\tau(0') = 0$ and $\tau(n) = n$ for $n \in \mathbb{Z}_{\geq 0}$.

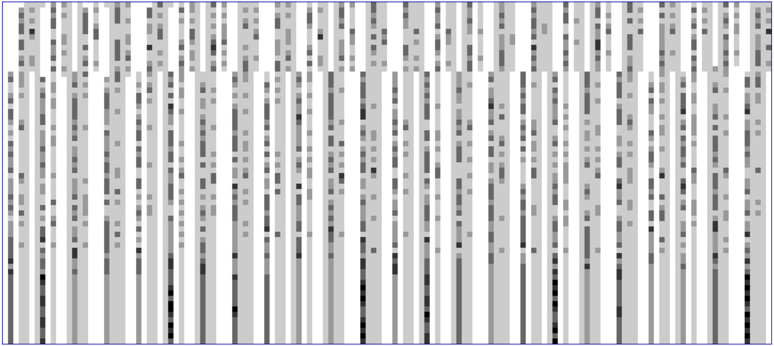
Theorem

There exist words u, v of lengths $|u| = 1001 - 1$ and $|v| = 29949$ such that $\mathbf{w}_{6/5} = \tau(\varphi^\infty(0'))$, where

$$\varphi(n) = \begin{cases} v\,\varphi(0) & \text{if } n = 0' \\ u\,(n+3) & \text{if } n \geq 0. \end{cases}$$

$\mathbf{w}_{5/4}$ wrapped into 144 columns

$$\mathbf{w}_{5/4} = 000011110202101001011212000013110102101302\cdots$$



We don't know the structure of $\mathbf{w}_{5/4}$.

Catalogue

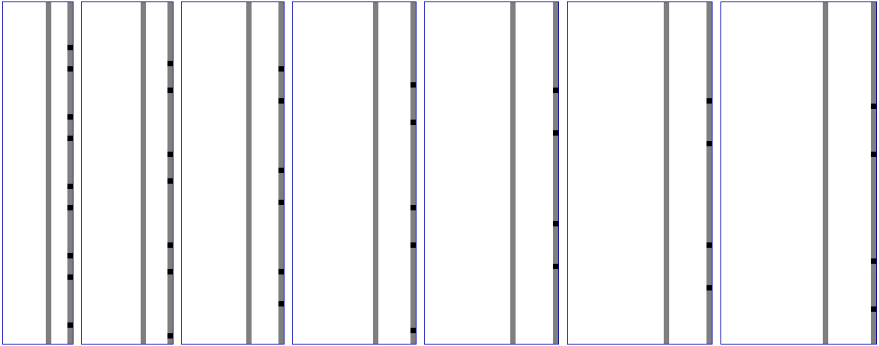
For many words $\mathbf{w}_{a/b}$, there is a related k-uniform morphism.
A k-uniform morphism generates a k-regular sequence.

$$\begin{aligned}
\tfrac{a}{b} = \tfrac{3}{2} \quad &\rightarrow \quad k = 6 \\
\tfrac{a}{b} = \tfrac{5}{3} \quad &\rightarrow \quad k = 7 \\
\tfrac{a}{b} = \tfrac{7}{4} \quad &\rightarrow \quad k = 50847 \\
\tfrac{a}{b} = \tfrac{8}{5} \quad &\rightarrow \quad k = 733 \\
\tfrac{a}{b} = \tfrac{6}{5} \quad &\rightarrow \quad k = 1001 \\
\tfrac{a}{b} = \tfrac{5}{4} \quad &\rightarrow \quad k = ?
\end{aligned}$$

Question

Is every $\mathbf{w}_{a/b}$ k-regular for some k? How is k related to $\frac{a}{b}$?

A family related to $\mathbf{w}_{5/3}$



$\mathbf{w}_{9/5}$ $\mathbf{w}_{12/7}$ $\mathbf{w}_{13/7}$ $\mathbf{w}_{16/9}$ $\mathbf{w}_{17/9}$ $\mathbf{w}_{19/11}$ $\mathbf{w}_{20/11}$

The interval $\frac{5}{3} \leq \frac{a}{b} < 2$

Theorem

Let $\frac{5}{3} \leq \frac{a}{b} < 2$ and b odd. Let φ be the $(2a - b)$-uniform morphism

$$\varphi(n) = 0^{a-1}\, 1\, 0^{a-b-1}\, (n+1)$$

for all $n \in \mathbb{Z}_{\geq 0}$. Then $\mathbf{w}_{a/b} = \varphi^\infty(0)$.

1. Show that φ preserves $\frac{a}{b}$-power-freeness.
   That is, if w is $\frac{a}{b}$-power-free then $\varphi(w)$ is $\frac{a}{b}$-power-free.
   Since 0 is $\frac{a}{b}$-power-free, it follows that $\varphi^\infty(0)$ is $\frac{a}{b}$-power-free.
2. Show that decrementing any letter in $\mathbf{w}_{a/b}$ introduces an $\frac{a}{b}$-power.

Other intervals

We have 30 symbolic $\frac{a}{b}$-power-free morphisms, found experimentally.

Theorem

Let $\frac{3}{2} < \frac{a}{b} < \frac{5}{3}$ *and* $\gcd(b, 5) = 1$. *The* $(5a - 4b)$*-uniform morphism*

$$\varphi(n) = 0^{a-1}\,1\,0^{a-b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,(n+1)$$

is $\frac{a}{b}$*-power-free.*

Theorem

Let $\frac{6}{5} < \frac{a}{b} < \frac{5}{4}$ *and* $\frac{a}{b} \notin \{\frac{11}{9}, \frac{17}{14}\}$. *The a-uniform morphism*

$$\varphi(n) = 0^{6a-7b-1}\,1\,0^{-3a+4b-1}\,1\,0^{-8a+10b-1}\,1\,0^{6a-7b-1}\,(n+1)$$

is $\frac{a}{b}$*-power-free.*

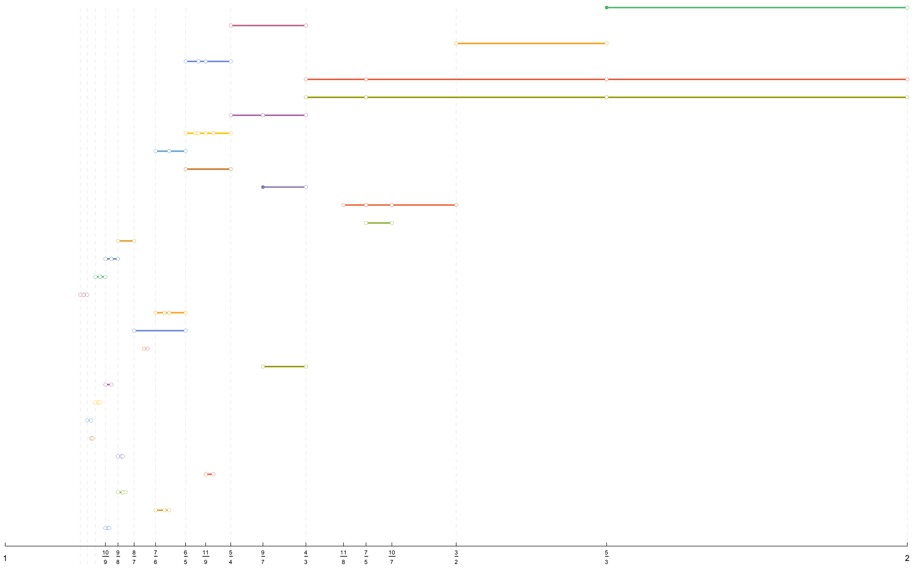
Other intervals

Theorem 50. *Let a, b be relatively prime positive integers such that $\frac{10}{9} < \frac{a}{b} < \frac{29}{26}$ and $\frac{a}{b} \neq \frac{39}{35}$ and $\gcd(b, 67) = 1$. Then the $(67a - 30b)$-uniform morphism*

$$\begin{aligned}
\varphi(n) = {}& 0^{-7a+8b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{-26a+29b-1}\,1\,0^{28a-31b-1}\,1\,0^{2a-2b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{2a-2b-1}\,1\,0^{11a-12b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{2a-2b-1}\,1\,0^{-24a+27b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{-24a+27b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1 \\
& 0^{11a-12b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{28a-31b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-7a+8b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{9a-10b-1}\,1\,0^{-7a+8b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{a-b-1}\,1\,0^{9a-10b-1}\,1 \\
& 0^{-7a+8b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{2a-2b-1}\,2\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{2a-2b-1}\,1 \\
& 0^{11a-12b-1}\,1\,0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,1\,0^{11a-12b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{11a-12b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{-24a+27b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-7a+8b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{28a-31b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{-26a+29b-1}\,1\,0^{28a-31b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{9a-10b-1}\,1\,0^{-7a+8b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1 \\
& 0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,2 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2 \\
& 0^{a-b-1}\,1\,0^{2a-2b-1}\,1\,0^{-24a+27b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{11a-12b-1}\,1\,0^{2a-2b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1 \\
& 0^{11a-12b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{11a-12b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-7a+8b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-7a+8b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1\,0^{a-b-1}\,1\,0^{9a-10b-1}\,(n+1),
\end{aligned}$$

with 279 nonzero letters, locates words of length $5a - 4b$ and is $\frac{a}{b}$-power-free.

Coverage of $\frac{a}{b}$-power-free morphisms

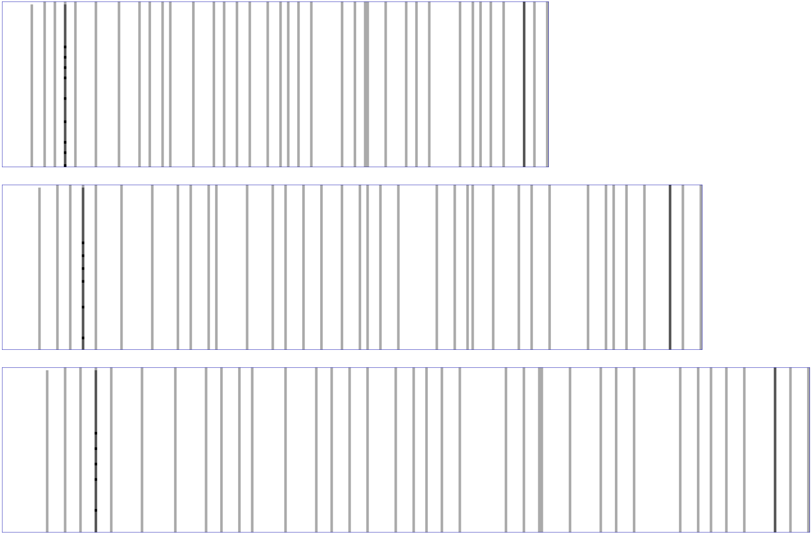


A family with a transient

$\mathbf{w}_{17/13}$

$\mathbf{w}_{22/17}$

$\mathbf{w}_{25/19}$



The interval $\frac{9}{7} < \frac{a}{b} < \frac{4}{3}$

Theorem

Let $\frac{9}{7} < \frac{a}{b} < \frac{4}{3}$ *and* $\gcd(b, 6) = 1$. *Let*

$$\varphi(0') = 0' 0^{a-2}\, 1\, 0^{a-b-1}\, 1\, 0^{a-b-1}\, 1 \varphi(0)$$

and

$$\begin{aligned}\varphi(n) = {} & 0^{a-b-1}\, 1\, 0^{2a-2b-1}\, 1\, 0^{-a+2b-1}\, 1\, 0^{2a-2b-1}\, 1\, 0^{a-b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{4a-5b-1}\, 1 \\ & 0^{-a+2b-1}\, 1\, 0^{2a-2b-1}\, 1\, 0^{a-b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{5a-6b-1}\, 1 \\ & 0^{-2a+3b-1}\, 1\, 0^{4a-5b-1}\, 1\, 0^{a-b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{3a-3b-1}\, 1\, 0^{-2a+3b-1}\, 1 \\ & 0^{a-b-1}\, 1\, 0^{-3a+4b-1}\, 1\, 0^{5a-6b-1}\, 1\, 0^{2a-2b-1}\, 1\, 0^{a-b-1}\, 1\, 0^{-2a+3b-1}\, 1 \\ & 0^{3a-3b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{4a-5b-1}\, 1\, 0^{a-b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{2a-2b-1}\, 2 \\ & 0^{a-b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{3a-3b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{a-b-1}\, 1\, 0^{a-b-1}\, (n+2),\end{aligned}$$

for $n \in \mathbb{Z}_{\geq 0}$. *Then* $\mathbf{w}_{a/b} = \tau(\varphi^\infty(0'))$.

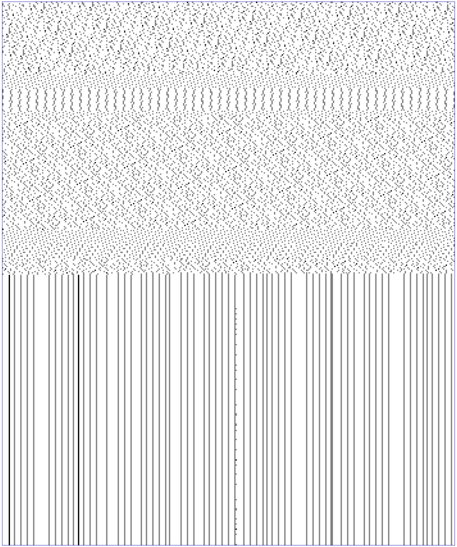
Sporadic rationals

The same proof technique applies to symbolic and explicit rationals. . .

The letters of $\mathbf{w}_{7/4}$ form a 50847-regular sequence.
The letters of $\mathbf{w}_{8/5}$ form a 733-regular sequence.
The letters of $\mathbf{w}_{13/9}$ form a 45430-regular sequence.
The letters of $\mathbf{w}_{17/10}$ form a 55657-regular sequence.
etc.

Is there some way to make sense of them?

$\mathbf{w}_{27/23}$ wrapped into 353 columns



There exist words u, v on $\{0, 1, 2\}$ of lengths $|u| = 353 - 1$ and $|v| = 75019$ such that $\mathbf{w}_{27/23} = \tau(\varphi^{\omega}(0'))$, where

$$\varphi(n) = \begin{cases} v\,\varphi(0) & \text{if } n = 0' \\ u\,(n+0) & \text{if } n \geq 0. \end{cases}$$

$\mathbf{w}_{27/23}$ is also the lex least $\frac{27}{23}$-power-free word on $\{0, 1, 2\}$.