Extremal words avoiding a fractional power

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Avoiding squares


Axel Thue (1863–1922)

A square is a nonempty word of the form xx. For example: $00, 0101$.
Are there arbitrarily long square-free words on the alphabet $\{0, 1\}$?

Try to construct one!

010☒

Avoiding squares

Are squares avoidable on the alphabet $\{0, 1, 2\}$?

$$010201202101201020120210201021012010201202 10\cdots$$

Theorem (Thue 1906)

There exist arbitrarily long square-free words on 3 letters.

The backtracking algorithm builds the lexicographically least word.

Open question (Allouche–Shallit, *Automatic Sequences* §1.10)

What is the structure of the lex. least square-free word on $\{0, 1, 2\}$?

Avoiding overlaps

An overlap is a word of the form xxc, where c is the first letter of x.
For example: $000, 01010$.

Overlaps are avoidable on a binary alphabet (Thue 1912).

The Thue–Morse morphism is defined by $\varphi(0) = 01$ and $\varphi(1) = 10$.

$$\begin{aligned}
\varphi(0) &= 01 \\
\varphi^2(0) &= \varphi(01) = 0110 \\
\varphi^3(0) &= \varphi(0110) = 01101001 \\
&\vdots
\end{aligned}$$

The Thue–Morse word

$$\varphi^\infty(0) = 01101001100101101001011001101001\cdots$$

is overlap-free.

Morphisms

A morphism on a set Σ is a function $\varphi \colon \Sigma^* \to \Sigma^*$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all words $x, y \in \Sigma^*$.

If there is a letter $c \in \Sigma$ such that $\varphi(c) = c\,x$ for some word x, then

$$\begin{aligned}
\varphi(c) &= c\,x \\
\varphi^2(c) &= \varphi(c\,x) = c\,x\,\varphi(x) \\
\varphi^3(c) &= \varphi(c\,x\,\varphi(x)) = c\,x\,\varphi(x)\,\varphi^2(x) \\
&\vdots \\
\varphi^\infty(c) &= c\,x\,\varphi(x)\,\varphi^2(x)\,\varphi^3(x)\,\varphi^4(x)\cdots.
\end{aligned}$$

$\varphi^\infty(c)$ is a fixed point of φ.

Uniform morphisms

φ is *k*-uniform if $|\varphi(c)| = k$ for all $c \in \Sigma$.
The Thue–Morse morphism is 2-uniform: $\varphi(0) = 01$, $\varphi(1) = 10$.

Fixed points of k-uniform morphisms reflect base-k representations.

Let $w(i)$ be the ith letter of the Thue–Morse word

$$\varphi^\infty(0) = 01101001100101101001011001101001 \cdots .$$

i	$\mathrm{rep}_2(i)$	$w(i)$
0	ϵ	0
1	1	1
2	10	1
3	11	0
4	100	1
5	101	0
6	110	0
7	111	1

Infinite alphabet

What is the lexicographically least square-free word on $\mathbb{Z}_{\geq 0}$?

$$010201030102010401020103010201 05\cdots$$

Theorem (Guay-Paquet–Shallit 2009)

Let $\varphi(n) = 0\,(n+1)$.
The lexicographically least square-free word on $\mathbb{Z}_{\geq 0}$ is $\varphi^\infty(0)$.

φ is 2-uniform.

$$\varphi(0) = 01$$
$$\varphi^2(0) = 0102$$
$$\varphi^3(0) = 01020103$$
$$\vdots$$

For each integer $a \geq 2$, let $\varphi(n) = 0^{a-1}(n+1)$.
The lexicographically least a-power-free word on $\mathbb{Z}_{\geq 0}$ is $\varphi^\infty(0)$.

Avoiding overlaps

What is the lexicographically least overlap-free word on $\mathbb{Z}_{\geq 0}$?

$$\underset{\varphi(0)}{001}\;\underset{\varphi(0)}{001}\;\underset{\varphi(1)}{1001002}\;\underset{\varphi(0)}{001}\;\underset{\varphi(0)}{001}\;\underset{\varphi(1)}{1001002}\;\underset{\varphi(1)}{1001002}\;\underset{\varphi(0)}{001}\;\underset{\varphi(0)}{001}\;\underset{\varphi(1)}{1001002}\;\underset{\varphi(0)}{001}\;\underset{\varphi(0)}{001}\;\underset{\varphi(2)}{2001001100100200100110010003}\;\underset{\cdots}{\cdots}$$

Let σ be the right shift: $\sigma(xc) = cx$ for words x and letters c.

Theorem (Guay-Paquet–Shallit 2009)

Define φ recursively by $\varphi(n) = \sigma(\varphi^n(00))\,(n+1)$.
The lexicographically least overlap-free word on $\mathbb{Z}_{\geq 0}$ is $\varphi^\infty(0)$.

φ is non-uniform.

$$\varphi(0) = 001$$
$$\varphi^2(0) = 0010011001002$$
$$\vdots$$

Fractional powers

$01220 = (0122)^{5/4}$ is a $\frac{5}{4}$-power.
$011101 = (0111)^{3/2}$ is a $\frac{3}{2}$-power.

Definition

Let $\frac{a}{b} > 1$. A word w is an $\frac{a}{b}$-power if

$$w = (xy)^e x$$

and $\frac{|w|}{|xy|} = \frac{a}{b}$ for some words x, y and some integer $e \geq 1$.

$\frac{5}{4}$-powers look like xyx where $|y| = 3|x|$.
$\frac{3}{2}$-powers look like xyx where $|y| = |x|$.

Notation

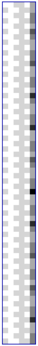
Let $\mathbf{w}_{a/b}$ be the lex. least $\frac{a}{b}$-power-free word on $\mathbb{Z}_{\geq 0}$.

We assume $\gcd(a, b) = 1$.

Avoiding $3/2$-powers

$$\mathbf{w}_{3/2} = 001102100112001103100113001102100114001103\cdots$$

$$\mathbf{w}_{3/2} = \begin{array}{l} 001102 \\ 100112 \\ 001103 \\ 100113 \\ 001102 \\ 100114 \\ 001103 \\ 100112 \\ \vdots \end{array}$$



Theorem (Rowland–Shallit 2012)

The ith letter $w(i)$ of $\mathbf{w}_{3/2}$ satisfies $w(6i+5) = w(i) + 2$.

Other notions of avoidance

Notation

- Let $\mathbf{w}_{\geq a/b}$ be the lex. least infinite word on $\mathbb{Z}_{\geq 0}$ avoiding $\frac{p}{q}$-powers for all $\frac{p}{q} \geq \frac{a}{b}$.
- Let $\mathbf{w}_{>a/b}$ be the lex. least infinite word on $\mathbb{Z}_{\geq 0}$ avoiding $\frac{p}{q}$-powers for all $\frac{p}{q} > \frac{a}{b}$.

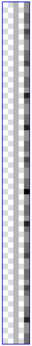
What are the relationships between $\mathbf{w}_{a/b}$, $\mathbf{w}_{\geq a/b}$, and $\mathbf{w}_{>a/b}$?

The lex. least overlap-free word is $\mathbf{w}_{>2}$.

Avoiding $\geq 3/2$-powers

$\mathbf{w}_{\geq 3/2} = 0120310213012041021401203102150120410213 01203\cdots$

$$\mathbf{w}_{\geq 3/2} = \begin{array}{l} 01203 \\ 10213 \\ 01204 \\ 10214 \\ 01203 \\ 10215 \\ 01204 \\ 10213 \\ \vdots \end{array}$$

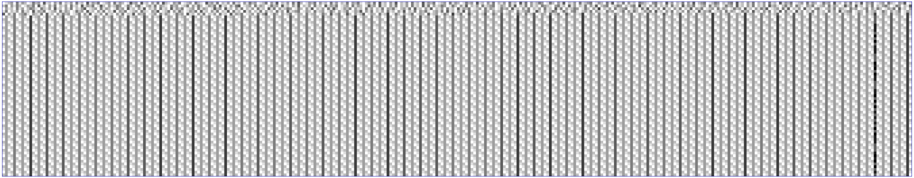


Theorem

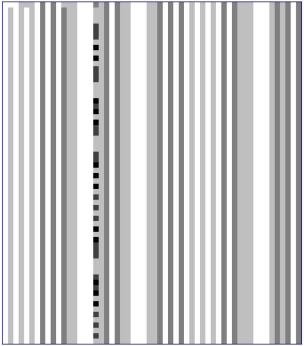
We have $\mathbf{w}_{\geq 3/2}(5i+4) = \mathbf{w}_{3/2}(i) + 3$ *for all* $i \geq 0$.

Avoiding 4/3-powers

$\mathbf{w}_{\geq 4/3}$:



$\mathbf{w}_{4/3}$:



Conjecture:
$\mathbf{w}_{\geq 4/3}(336i+1666) = \mathbf{w}_{4/3}(56i+17)+4$ for all $i \geq 0$.

Are there similar relationships between $\mathbf{w}_{\geq a/b}$ and $\mathbf{w}_{a/b}$ for other $\frac{a}{b}$?

We focus on $\mathbf{w}_{a/b}$.

The interval $\frac{a}{b} \geq 2$

$$\mathbf{w}_{5/2} = 00001000010000100001000020000100001\cdots = \mathbf{w}_5 = \varphi^\infty(0)$$

where $\varphi(n) = 0000(n+1)$.

Theorem

If $\frac{a}{b} \geq 2$, then $\mathbf{w}_{a/b} = \mathbf{w}_a$.

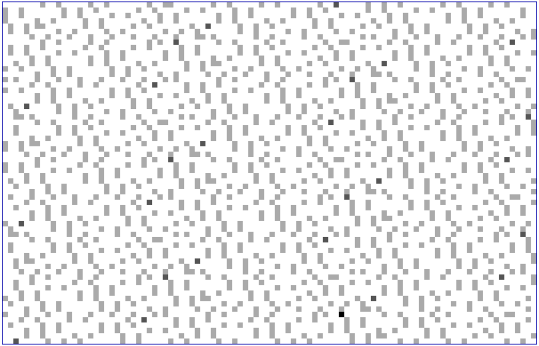
Proof (one direction).

The a-power $v^a = (v^b)^{a/b}$ is also an $\frac{a}{b}$-power.
So $\mathbf{w}_{a/b}$ is a-power-free. Thus $\mathbf{w}_a \leq \mathbf{w}_{a/b}$ lexicographically. □

Therefore it suffices to consider $1 < \frac{a}{b} < 2$.

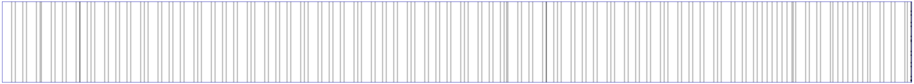
$\mathbf{w}_{8/5}$ wrapped into 100 columns

$$\mathbf{w}_{8/5} = 000000010010000010010000000100110000000100 \cdots$$



$\mathbf{w}_{8/5}$ wrapped into 733 columns

$$\mathbf{w}_{8/5} = 00000001001000000100100000000100110000000100\cdots$$



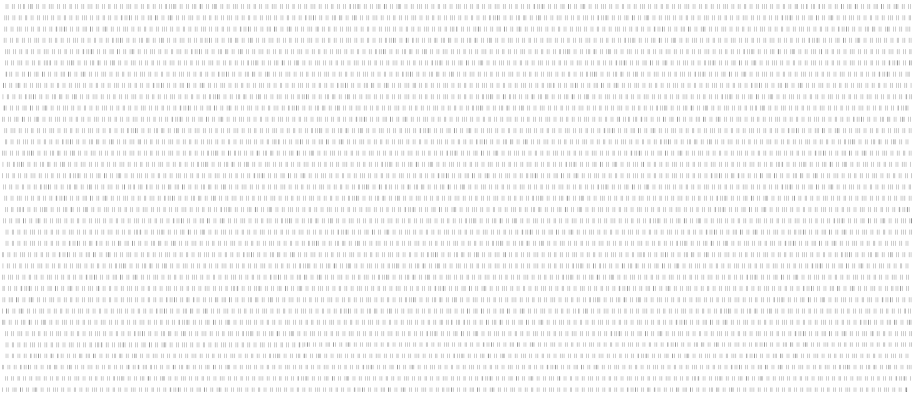
Theorem

$\mathbf{w}_{8/5} = \varphi^\infty(0)$ *for the 733-uniform morphism*

$$\begin{aligned}\varphi(n) = {} & 00000001001000000100100000000100110000000100100000010010000000010020000 \\ & 01001001000000001001000000100100000010010000000010010010000000100100000 \\ & 10010000001001000000001001001000000001001000000100100000010010000000001001 \\ & 00100000000100100000010010000001001000000001001001000000001001000000010010 \\ & 00001001000000001001001000000001001000000100100000010010000000100100100 \\ & 00000100100000010010000001001000000001001001000000001001000000100100000001 \\ & 00101100000000100100000010010000000100200000010010010000000010010000010 \\ & 01000000100100000000100100100000000100100000010010000001001000000000100100 \\ & 10000000100100000010010000001001000000001001001000000001001000000100100000 \\ & 00100100010001000100010001000100011010000000010010000001001000000000101 \\ & 000100010001000100010001000101000000001001000000100100000000010100(n+2).\end{aligned}$$

$\mathbf{w}_{7/4}$ wrapped into 50847 columns

$$\mathbf{w}_{7/4} = 000000100100000001001000000010010000011000000\cdots$$

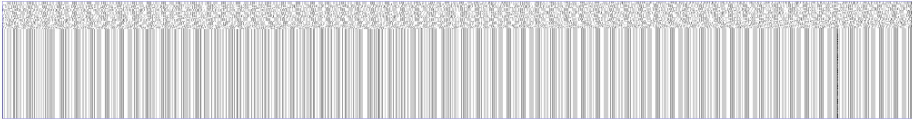


Theorem

$\mathbf{w}_{7/4} = \varphi^{\infty}(0)$ *for some* 50847*-uniform morphism* $\varphi(n) = u(n+2)$.

$\mathbf{w}_{6/5}$ wrapped into 1001 columns

$$\mathbf{w}_{6/5} = 000001111102020201011101000202120210110010\cdots$$



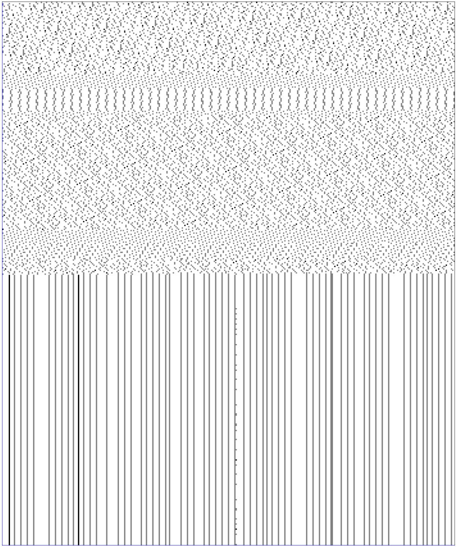
There is a transient region.

Introduce a new letter $0'$, and let $\tau(0') = 0$ and $\tau(n) = n$ for $n \in \mathbb{Z}_{\geq 0}$.

There exist words u, v of lengths $|u| = 1000$ and $|v| = 29949$ such that $\mathbf{w}_{6/5} = \tau(\varphi^\infty(0'))$, where

$$\varphi(n) = \begin{cases} v\,\varphi(0) & \text{if } n = 0' \\ u\,(n+2) & \text{if } n \in \mathbb{Z}. \end{cases}$$

$\mathbf{w}_{27/23}$ wrapped into 353 columns



There exist words u, v on $\{0,1,2\}$ of lengths $|u| = 352$ and $|v| = 75019$ such that $\mathbf{w}_{27/23} = \tau(\varphi^\omega(0'))$, where

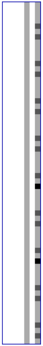
$$\varphi(n) = \begin{cases} v\,\varphi(0) & \text{if } n = 0' \\ u\,(n + 0) & \text{if } n \in \mathbb{Z}. \end{cases}$$

$\mathbf{w}_{27/23}$ is also the lex. least $\frac{27}{23}$-power-free word on $\{0,1,2\}$.

$\mathbf{w}_{5/3}$

$$\mathbf{w}_{5/3} = 0000101000010100001010000101000010200001 01\cdots$$

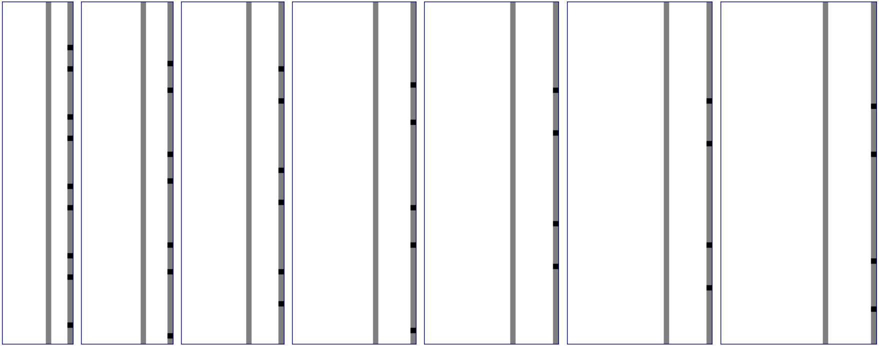
$$\begin{aligned}\mathbf{w}_{5/3} = {} & 0000101 \\ & 0000101 \\ & 0000101 \\ & 0000101 \\ & 0000102 \\ & 0000101 \\ & 0000102 \\ & 0000101 \\ & \vdots\end{aligned}$$



$$w(7i+6) = w(i) + 1$$

$\mathbf{w}_{5/3} = \varphi^{\infty}(0)$, where $\varphi(n) = 000010(n+1)$ is a 7-uniform morphism.

A family related to $\mathbf{w}_{5/3}$



Theorem (Pudwell–Rowland 2018)

Let $\frac{5}{3} \leq \frac{a}{b} < 2$ with b odd. Then $\mathbf{w}_{a/b} = \varphi^\infty(0)$, where $\varphi(n) = 0^{a-1}\,1\,0^{a-b-1}\,(n+1)$ is a $(2a-b)$-uniform morphism.

Catalog of $\mathbf{w}_{a/b}$

General recurrence for self-similar column: $w(ki + r') = w(i + s) + d(i)$.

a/b	k	$d(i)$	r'	s	rank	note
$a \in \mathbb{Z}_{\geq 2}$	a	1	0	0	2	
$3/2$	6	2	0	0	3	
$4/3$	56	$1, 2$	73	0	4	
$5/3$	7	1	0	0	2	
$5/4$	6	$1, 2, 3$	123061	5920	188	
$7/4$	50847	2	0	0	2	
$6/5$	1001	3	30949	0	33	
$7/5$	80874	1	173978	0		conjectural
$8/5$	733	2	0	0	2	
$9/5$	13	1	0	0	2	
$7/6$	41190	3	41201	0		conjectural
$11/6$						[no conjecture]

Does every word $\mathbf{w}_{a/b}$ arise from some k-uniform morphism?

References

- Mathieu Guay-Paquet and Jeffrey Shallit, Avoiding squares and overlaps over the natural numbers, *Discrete Mathematics* **309** (2009) 6245–6254.
- Lara Pudwell and Eric Rowland, Avoiding fractional powers over the natural numbers, *The Electronic Journal of Combinatorics* **25** (2018) #P2.27.
- Eric Rowland and Jeffrey Shallit, Avoiding $3/2$-powers over the natural numbers, *Discrete Mathematics* **312** (2012) 1282–1288.
- Eric Rowland and Manon Stipulanti, Avoiding $5/4$-powers on the alphabet of non-negative integers, *The Electronic Journal of Combinatorics* **27** (2020) #P3.42.