Extremal sequences avoiding a fractional power

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joint work with Jeff Shallit and Lara Pudwell

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Periodic sequences

Periodic sequences are simple.

$$000000000000000000000000000000000000000000 \cdots$$
$$010201020102010201020102010201020102010201 \cdots$$
$$011011011011011011011011011011011011011011 \cdots$$

Natural vague question:
What are the simplest non-periodic sequences?

A square is a nonempty word of the form *ww*.

Squares on a 2-letter alphabet

Axel Thue (1863–1922)

Are squares avoidable on a 2-letter alphabet?
Are there arbitrarily long square-free words on $\{0, 1\}$?

Choose an order on $\{0, 1\}$ and try to construct one:

$$010\boxed{X}$$

Squares on a 3-letter alphabet

Are squares avoidable on $\{0, 1, 2\}$?

$$01020102\boxtimes 0210120102012021020102101201020120210\cdots$$

Theorem (Thue 1906)

There exist arbitrarily long square-free words on 3 *letters.*

The backtracking algorithm builds the lexicographically least sequence.

Open problem (Allouche–Shallit, *Automatic Sequences* §1.10)

Characterize the lex. least square-free sequence on $\{0, 1, 2\}$.

Infinite alphabet

On an infinite alphabet, the backtracking algorithm doesn't backtrack.

Are squares avoidable on $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$? Yes.

$$0102010301020104010201030102010 5\cdots$$

Theorem (Guay-Paquet–Shallit 2009)

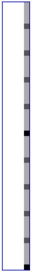
Let $\varphi(n) = 0\,(n+1)$.
The lexicographically least square-free sequence on $\mathbb{Z}_{\geq 0}$ is $\varphi^\infty(0)$.

$$\begin{aligned}
\varphi(0) &= 01 \\
\varphi^2(0) &= 0102 \\
\varphi^3(0) &= 01020103 \\
&\vdots \\
\varphi^\infty(0) &= 01020103010201040102010301020105\cdots
\end{aligned}$$

More generally, let $a \geq 2$. Let $\varphi(n) = 0^{a-1}(n+1)$.
The lexicographically least a-power-free sequence on $\mathbb{Z}_{\geq 0}$ is $\varphi^\infty(0)$.

$$\mathbf{s}_5 = 000010000100001000010000200001000001 \cdots$$

$$\begin{array}{rl} \mathbf{s}_5 = & 00001 \\ & 00001 \\ & 00001 \\ & 00001 \\ & 00002 \\ & 00001 \\ & \vdots \end{array}$$



$$\begin{aligned} s(5n+0) &= 0 \\ s(5n+1) &= 0 \\ s(5n+2) &= 0 \\ s(5n+3) &= 0 \\ s(5n+4) &= s(n)+1 \end{aligned}$$

$\mathbf{s}_5$ satisfies a recurrence reflecting the base-5 representation of n.
Such a sequence is called 5-regular.

Fractional powers

$011101 = (0111)^{3/2}$ is a $\frac{3}{2}$-power.
If $|x| = |y| = |z|$, then $xyzxyzx = (xyz)^{7/3}$ is a $\frac{7}{3}$-power.

Definition

A word w is an $\frac{a}{b}$-power if

$$w = v^e x$$

where $e \geq 0$ is an integer, x is a prefix of v, and $\frac{|w|}{|v|} = \frac{a}{b}$.

Notation

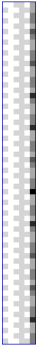
For $\frac{a}{b} > 1$, let $\mathbf{s}_{a/b}$ be the lex. least $\frac{a}{b}$-power-free sequence on $\mathbb{Z}_{\geq 0}$.

We assume $\gcd(a, b) = 1$ from now on.

Avoiding $3/2$-powers

$$\mathbf{s}_{3/2} = 001102100112001103100113001102100114001103\cdots$$

$$\begin{aligned}\mathbf{s}_{3/2} = \; & 001102\\ & 100112\\ & 001103\\ & 100113\\ & 001102\\ & 100114\\ & 001103\\ & 100112\\ & \vdots\end{aligned}$$



$$s(6n+5) = s(n) + 2$$

Theorem (Rowland–Shallit 2012)

The sequence $\mathbf{s}_{3/2}$ is 6-regular.

Why 6?

k-regular sequences

An integer sequence $s(n)_{n\geq 0}$ is k-regular if the set

$$\left\{s(k^e n + r)_{n\geq 0} : e \geq 0 \text{ and } 0 \leq r \leq k^e - 1\right\}$$

is contained in a finite-dimensional $\mathbb{Q}$-vector space.

Analogously: $s(n)_{n\geq 0}$ is constant-recursive if $\{s(n+r)_{n\geq 0} : r \geq 0\}$ is contained in a finite-dimensional $\mathbb{Q}$-vector space.

Is the value of k unique?
No; a 2-regular sequence is also 4-regular, and vice versa.

But almost: If k and ℓ are multiplicatively independent and $s(n)_{n\geq 0}$ is both k-regular and ℓ-regular, then $\sum_{n\geq 0} s(n)x^n$ is the power series of a rational function whose poles are roots of unity [Bell 2006].

So the value of k gives structural information.

The interval $\frac{a}{b} \geq 2$

$$\mathbf{s}_{5/2} = 00001000010000100001000020000100001 \cdots = \mathbf{s}_5$$

Theorem

If $\frac{a}{b} \geq 2$, then $\mathbf{s}_{a/b} = \mathbf{s}_a$.

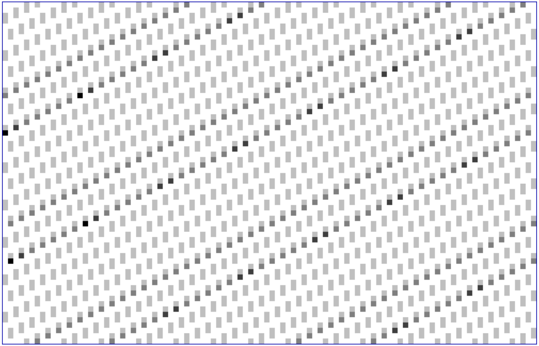
Proof (one direction).

Every a-power $v^a = (v^b)^{a/b}$ is also an $\frac{a}{b}$-power.
So $\mathbf{s}_{a/b}$ is a-power-free. Thus $\mathbf{s}_a \leq \mathbf{s}_{a/b}$ lexicographically. □

It suffices to consider $1 < \frac{a}{b} < 2$.

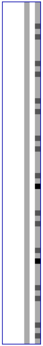
$\mathbf{s}_{5/3}$ wrapped into 100 columns

$$\mathbf{s}_{5/3} = 000010100001010000101000010100001020000101 \cdots$$



$\mathbf{s}_{5/3}$ wrapped into 7 columns

$$\mathbf{s}_{5/3} = 000010100001010000101000010100001020000101\cdots$$

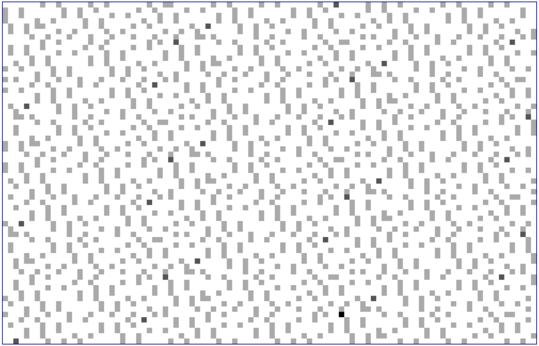


Theorem

$\mathbf{s}_{5/3} = \varphi^\infty(0)$, *where* $\varphi(n) = 000010(n+1)$ *is a* 7*-uniform morphism.*

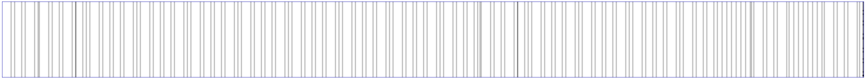
$\mathbf{s}_{8/5}$ wrapped into 100 columns

$$\mathbf{s}_{8/5} = 00000001001000001001000000001001100000000100\cdots$$



$\mathbf{s}_{8/5}$ wrapped into 733 columns

$$\mathbf{s}_{8/5} = 00000001001000000100100000000100110000000100\cdots$$



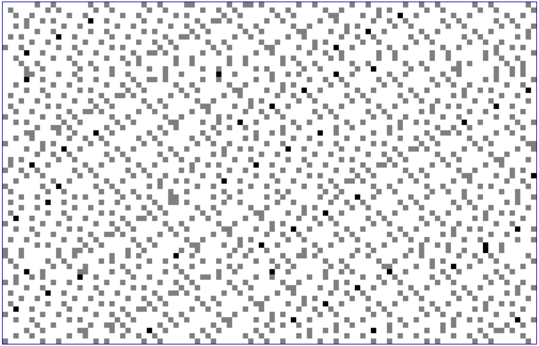
Theorem

$\mathbf{s}_{8/5} = \varphi^{\infty}(0)$ *for the* 733*-uniform morphism*

$$
\begin{aligned}
\varphi(n) = {} & 000000010010000001001000000001001100000000100100000010010000000010020000 \\
& 010010010000000010010000001001000000100100000000100100100000000100100000 \\
& 100100000010010000000010010010000000010010000001001000000100100000001001 \\
& 001000000001001000000100100000010010000000010010010000000010010000010010 \\
& 000010010000000010010010000000010010000001001000000100100000000100100100 \\
& 000001001000000100100000010010000000010010010000000010010000001001000001 \\
& 001011000000001001000000100100000000100200000010010010000000010010000010 \\
& 010000001001000000001001001000000001001000000100100000010010000000100100 \\
& 100000001001000000100100000010010000000010010010000000010010000001001000 \\
& 001001000100010001000100010001000110100000000100100000010010000000101 \\
& 000100010001000100010001000101000000001001000000100100000000010100(n+2).
\end{aligned}
$$

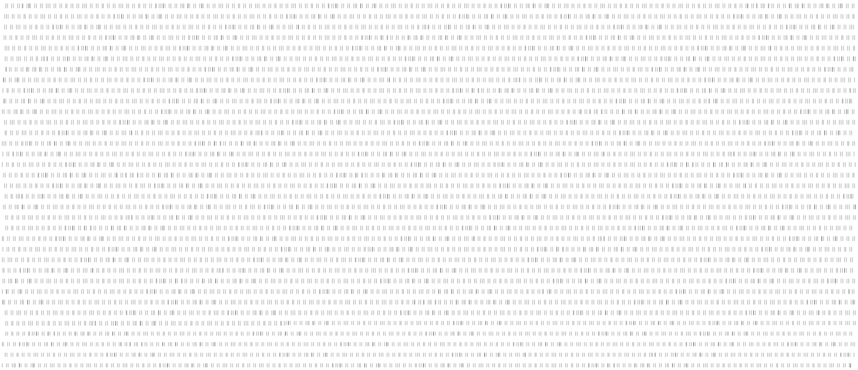
$\mathbf{s}_{7/4}$ wrapped into 100 columns

$$\mathbf{s}_{7/4} = 0000001001000000010010000000100100000110000000\cdots$$



$\mathbf{s}_{7/4}$ wrapped into 50847 columns

$$\mathbf{s}_{7/4} = 000000100100000001001000000010010000011000000\cdots$$

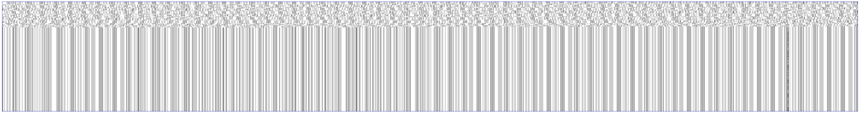


Theorem

$\mathbf{s}_{7/4} = \varphi^{\infty}(0)$ *for some* 50847*-uniform morphism* $\varphi(n) = u\,(n+2)$*.*

$\mathbf{s}_{6/5}$ wrapped into 1001 columns

$$\mathbf{s}_{6/5} = 000001111102020201011101000202120210110010\cdots$$



Introduce a new letter $0'$.
Let $\tau(0') = 0$ and $\tau(n) = n$ for $n \in \mathbb{Z}_{\geq 0}$.

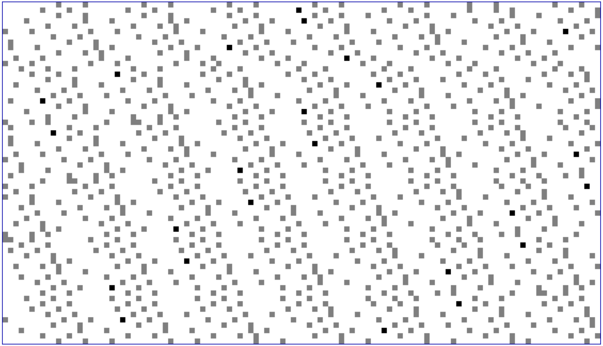
Theorem

There exist words u, v of lengths $|u| = 1001 - 1$ and $|v| = 29949$ such that $\mathbf{s}_{6/5} = \tau(\varphi^\infty(0'))$, where

$$\varphi(n) = \begin{cases} v\,\varphi(0) & \text{if } n = 0' \\ u\,(n+3) & \text{if } n \geq 0. \end{cases}$$

$\mathbf{s}_{11/6}$ wrapped into 112 columns

$$\mathbf{s}_{11/6} = 000000000010000100000000001000010000000000 \cdots$$



We don't know the structure of $\mathbf{s}_{11/6}$.

Catalogue

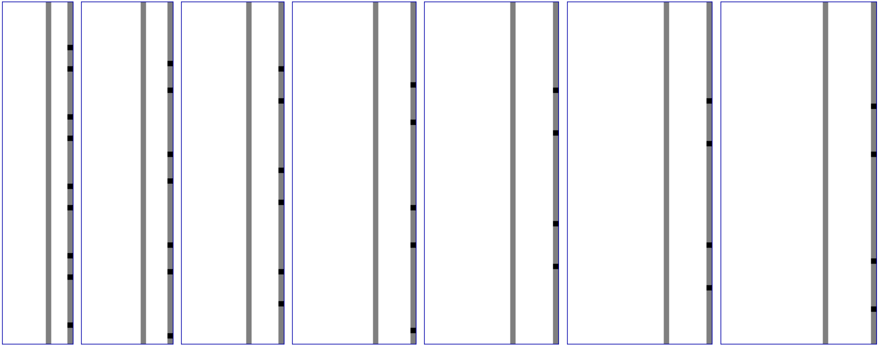
For many sequences $\mathbf{s}_{a/b}$, there is a related k-uniform morphism.
A k-uniform morphism generates a k-regular sequence.

$\frac{a}{b}$	k
$\frac{3}{2}$	6
$\frac{5}{3}$	7
$\frac{8}{5}$	733
$\frac{7}{4}$	50847
$\frac{6}{5}$	1001
$\frac{11}{6}$	?

Question

Is every $\mathbf{s}_{a/b}$ k-regular for some k? How is k related to $\frac{a}{b}$?

A family related to $\mathbf{s}_{5/3}$



$\mathbf{s}_{9/5}$ $\mathbf{s}_{12/7}$ $\mathbf{s}_{13/7}$ $\mathbf{s}_{16/9}$ $\mathbf{s}_{17/9}$ $\mathbf{s}_{19/11}$ $\mathbf{s}_{20/11}$

The interval $\frac{5}{3} \leq \frac{a}{b} < 2$

Theorem

Let $\frac{5}{3} \leq \frac{a}{b} < 2$ and b odd. Let φ be the $(2a - b)$-uniform morphism

$$\varphi(n) = 0^{a-1}\, 1\, 0^{a-b-1}\, (n+1)$$

for all $n \in \mathbb{Z}_{\geq 0}$. Then $\mathbf{s}_{a/b} = \varphi^\infty(0)$.

1. Show that φ preserves $\frac{a}{b}$-power-freeness.
   That is, if w is $\frac{a}{b}$-power-free then $\varphi(w)$ is $\frac{a}{b}$-power-free.
   Since 0 is $\frac{a}{b}$-power-free, it follows that $\varphi^\infty(0)$ is $\frac{a}{b}$-power-free.
2. Show that decrementing any term in $\mathbf{s}_{a/b}$ introduces an $\frac{a}{b}$-power.

Other intervals

We have 30 symbolic $\frac{a}{b}$-power-free morphisms, found experimentally.

Theorem

Let $\frac{3}{2} < \frac{a}{b} < \frac{5}{3}$ and $\gcd(b, 5) = 1$. The $(5a - 4b)$-uniform morphism

$$\varphi(n) = 0^{a-1}\,1\,0^{a-b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,(n+1)$$

is $\frac{a}{b}$-power-free.

Theorem

Let $\frac{6}{5} < \frac{a}{b} < \frac{5}{4}$ and $\frac{a}{b} \notin \{\frac{11}{9}, \frac{17}{14}\}$. The a-uniform morphism

$$\varphi(n) = 0^{6a-7b-1}\,1\,0^{-3a+4b-1}\,1\,0^{-8a+10b-1}\,1\,0^{6a-7b-1}\,(n+1)$$

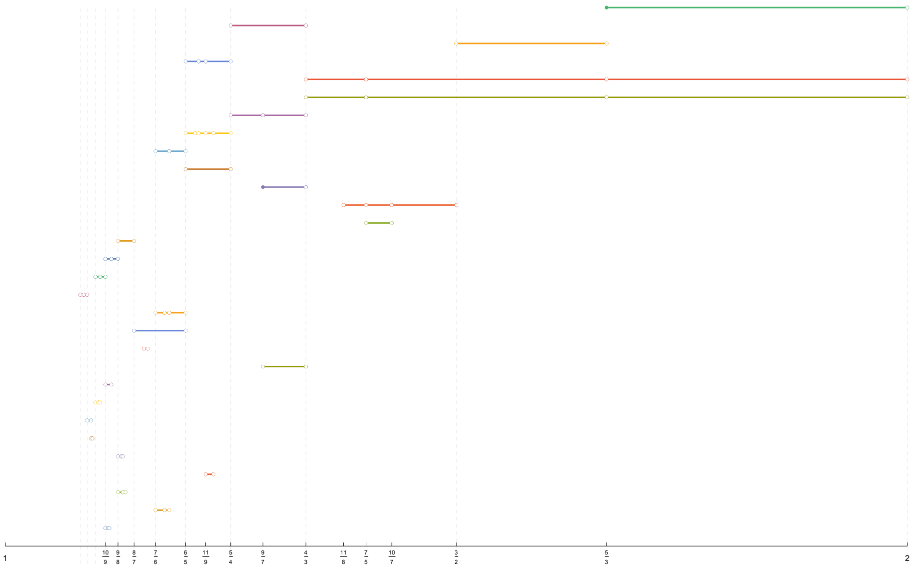
is $\frac{a}{b}$-power-free.

Theorem 50. *Let a, b be relatively prime positive integers such that $\frac{10}{9} < \frac{a}{b} < \frac{29}{26}$ and $\frac{a}{b} \neq \frac{39}{35}$ and $\gcd(b, 67) = 1$. Then the $(67a - 30b)$-uniform morphism*

$$
\begin{aligned}
\varphi(n) = {} & 0^{-7a+8b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{-26a+29b-1}\,1\,0^{28a-31b-1}\,1\,0^{2a-2b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{2a-2b-1}\,1\,0^{11a-12b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{2a-2b-1}\,1\,0^{-24a+27b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{-24a+27b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1 \\
& 0^{11a-12b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{28a-31b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-7a+8b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{9a-10b-1}\,1\,0^{-7a+8b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{a-b-1}\,1\,0^{9a-10b-1}\,1 \\
& 0^{-7a+8b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{2a-2b-1}\,2\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{2a-2b-1}\,1 \\
& 0^{11a-12b-1}\,1\,0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,1\,0^{11a-12b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{11a-12b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{-24a+27b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-7a+8b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{28a-31b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{-26a+29b-1}\,1\,0^{28a-31b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{9a-10b-1}\,1\,0^{-7a+8b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1 \\
& 0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,2 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2 \\
& 0^{a-b-1}\,1\,0^{2a-2b-1}\,1\,0^{-24a+27b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{11a-12b-1}\,1\,0^{2a-2b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1 \\
& 0^{11a-12b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{11a-12b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-7a+8b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-7a+8b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1\,0^{a-b-1}\,1\,0^{9a-10b-1}\,(n+1),
\end{aligned}
$$

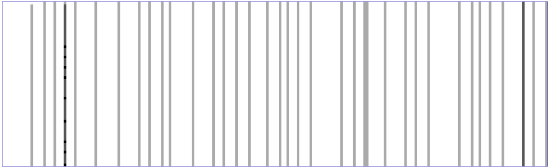
with 279 nonzero letters, locates words of length $5a - 4b$ and is $\frac{a}{b}$-power-free.

Coverage of $\frac{a}{b}$-power-free morphisms

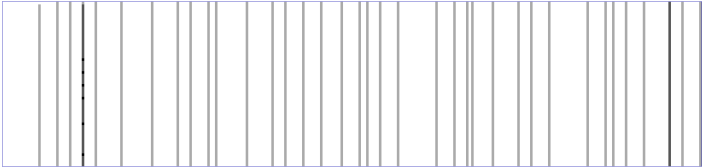


A family with a transient

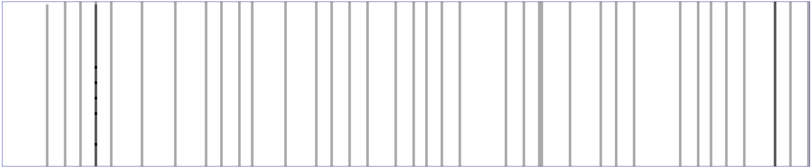
$\mathbf{s}_{17/13}$



$\mathbf{s}_{22/17}$



$\mathbf{s}_{25/19}$



The interval $\frac{9}{7} < \frac{a}{b} < \frac{4}{3}$

Theorem

Let $\frac{9}{7} < \frac{a}{b} < \frac{4}{3}$ *and* $\gcd(b, 6) = 1$. *Let*

$$\varphi(0') = 0'0^{a-2}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,1\varphi(0)$$

and

$$\begin{aligned}\varphi(n) = {} & 0^{a-b-1}\,1\,0^{2a-2b-1}\,1\,0^{-a+2b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{4a-5b-1}\,1 \\ & 0^{-a+2b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{-2a+3b-1}\,1\,0^{5a-6b-1}\,1 \\ & 0^{-2a+3b-1}\,1\,0^{4a-5b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{3a-3b-1}\,1\,0^{-2a+3b-1}\,1 \\ & 0^{a-b-1}\,1\,0^{-3a+4b-1}\,1\,0^{5a-6b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1 \\ & 0^{3a-3b-1}\,1\,0^{-2a+3b-1}\,1\,0^{4a-5b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{2a-2b-1}\,2 \\ & 0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{3a-3b-1}\,1\,0^{-2a+3b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,(n+2),\end{aligned}$$

for $n \in \mathbb{Z}_{\geq 0}$. *Then* $\mathbf{s}_{a/b} = \tau(\varphi^\infty(0'))$.

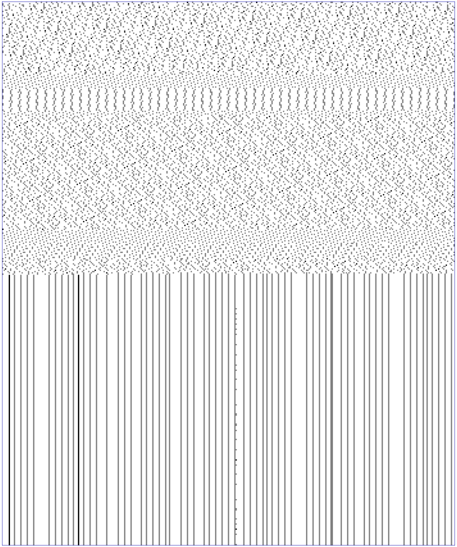
Sporadic rationals

The same proof technique applies to symbolic and explicit rationals...

$\mathbf{s}_{8/5}$ is a 733-regular sequence.
$\mathbf{s}_{7/4}$ is a 50847-regular sequence.
$\mathbf{s}_{13/9}$ is a 45430-regular sequence.
$\mathbf{s}_{17/10}$ is a 55657-regular sequence.
etc.

Is there some way to understand these values?

$\mathbf{s}_{27/23}$ wrapped into 353 columns



There exist words u, v on $\{0, 1, 2\}$ of lengths $|u| = 353 - 1$ and $|v| = 75019$ such that $\mathbf{s}_{27/23} = \tau(\varphi^{\infty}(0'))$, where

$$\varphi(n) = \begin{cases} v\,\varphi(0) & \text{if } n = 0' \\ u\,(n + 0) & \text{if } n \geq 0. \end{cases}$$

$$s(353n + 75371) = s(n)$$

$\mathbf{s}_{27/23}$ is a sequence on a finite alphabet!