Enumeration of binomial coefficients by their p-adic valuations

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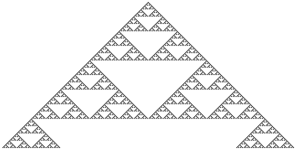
Odd binomial coefficients

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1



Glaisher (1899): How many odd entries are on each row?

$1, 2, 2, 4, 2, 4, 4, 8, 2, 4, 4, 8, 4, 8, 8, 16, \ldots \qquad 2^{|n|_1}$

$|n|_d :=$ number of occurrences of d in the base-p representation of n.

Proof

$\nu_p(n) := \max\{e \geq 0 : p^e \text{ divides } n\}$.
Example: $\nu_2(56) = 3$.

Kummer's theorem

$\nu_p(\binom{n}{m})$ = # *carries involved in adding* m *to* $n - m$ *in base* p.

Example

$n = 25$. How many m satisfy $\nu_2(\binom{25}{m}) = 0$?

$$n = 25 = 11001_2$$
$$m = **00*_2$$

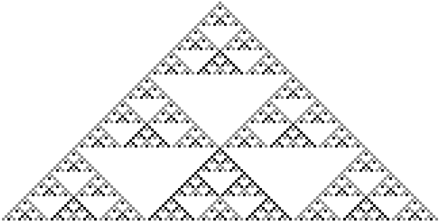
$2^{|25|_1} = 8$.

Fine's theorem

Theorem (Fine 1947)

Write $n = n_\ell \cdots n_1 n_0$ in base p. Then

$$\left|\left\{m : \binom{n}{m} \text{ is not divisible by } p\right\}\right| = (n_0+1)(n_1+1)\cdots(n_\ell+1)$$
$$= 1^{|n|_0} 2^{|n|_1} 3^{|n|_2} \cdots p^{|n|_{p-1}}.$$



Number of binomial coefficients not divisible by 3:

$1, 2, 3, 2, 4, 6, 3, 6, 9, 2, 4, 6, 4, 8, 12, 6, \ldots \qquad 2^{|n|_1} 3^{|n|_2}$

Prime powers?

Carlitz found a recurrence involving
$\theta_{p,\alpha}(n) := \left|\{m : 0 \leq m \leq n \text{ and } \nu_p(\binom{n}{m}) = \alpha\}\right|$ and
$\psi_{p,\alpha}(n) := \left|\{m : 0 \leq m \leq n \text{ and } \nu_p((m+1)\binom{n}{m}) = \alpha\}\right|$.

Theorem (Carlitz 1967)

$$\theta_{p,\alpha}(pn + d) = (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1)$$

$$\psi_{p,\alpha}(pn + d) = \begin{cases} (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1) & \text{if } 0 \leq d \leq p-2 \\ p\psi_{p,\alpha-1}(n) & \text{if } d = p-1. \end{cases}$$

Is there a formulation that treats digits uniformly?
And looks more like Fine's product?

Generating function

$$\textcolor{red}{T_p(n,x)} := \sum_{m=0}^{n} x^{\nu_p(\binom{n}{m})}$$

The coefficient of x^α is the number of $\binom{n}{m}$ with p-adic valuation α.

$n = 8$:	1	8	28	56	70	56	28	8	1
$\nu_2(\binom{8}{m})$:	0	3	2	3	1	3	2	3	0

n	$T_2(n,x)$
0	1
1	2
2	$x+2$
3	4
4	$2x^2+x+2$
5	$2x+4$
6	x^2+2x+4
7	8
8	$\textcolor{red}{4}x^3+\textcolor{red}{2}x^2+\textcolor{red}{1}x+\textcolor{red}{2}$

Matrix product

Theorem (Rowland 2018)

Write $n = n_\ell \cdots n_1 n_0$ in base p. Then

$$T_p(n,x) := \sum_{m=0}^{n} x^{\nu_p(\binom{n}{m})} = \begin{bmatrix} 1 & 0 \end{bmatrix} M_p(n_0)\, M_p(n_1) \, \cdots \, M_p(n_\ell) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$M_p(d) := \begin{bmatrix} d+1 & p-d-1 \\ d\,x & (p-d)\,x \end{bmatrix}$$

Example

$p = 2$, $n = 8 = 1000_2$:

$$\begin{aligned} T_2(8,x) &= \begin{bmatrix} 1 & 0 \end{bmatrix} M_2(0) M_2(0) M_2(0) M_2(1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2\,x \end{bmatrix}^3 \begin{bmatrix} 2 & 0 \\ x & x \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= 4x^3 + 2x^2 + x + 2. \end{aligned}$$

Comparison of recurrences

Carlitz recurrence:

$$\theta_{p,\alpha}(pn+d) = (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1)$$

$$\psi_{p,\alpha}(pn+d) = \begin{cases} (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1) & \text{if } 0 \le d \le p-2 \\ p\psi_{p,\alpha-1}(n) & \text{if } d = p-1. \end{cases}$$

Carlitz has $\psi_{p,\alpha}(pn+d)$ on the left but $\psi_{p,\alpha-1}(n-1)$ on the right.

Recurrence leading to matrix product:

$$\theta_{p,\alpha}(pn+d) = (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1)$$
$$\psi_{p,\alpha}(pn+d-1) = d\theta_{p,\alpha}(n) + (p-d)\psi_{p,\alpha-1}(n-1).$$

$$M_p(d) = \begin{bmatrix} d+1 & p-d-1 \\ d\,x & (p-d)\,x \end{bmatrix}$$

k-regularity

Definition (Allouche–Shallit 1992)

Let $k \geq 2$.
A sequence $s(n)_{n \geq 0}$ is *k-regular* if

$$\{s(k^e n + i)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq i \leq k^e - 1\}$$

is contained in a finite-dimensional vector space.

Compare to:

Definition

$s(n)_{n \geq 0}$ is *constant-recursive* if $\{s(n+i)_{n \geq 0} : i \geq 0\}$ is contained in a finite-dimensional vector space. Equivalently:

- $s(n)$ satisfies a linear recurrence involving $s(n+i)$
- $s(n) = u\, M^n\, v$ for some matrix M and vectors u, v
- the generating function $\sum_{n \geq 0} s(n) x^n$ is rational

Guessing a constant-recursive sequence

$\langle \{ s(n+i)_{n \geq 0} : i \geq 0 \} \rangle$ is finite-dimensional.

$s(n) = 2^n + n$:

$s(n)$:	$1, 3, 6, 11, 20, 37, \ldots$	basis element!
$s(n+1)$:	$3, 6, 11, 20, 37, 70, \ldots$	basis element!
$s(n+2)$:	$6, 11, 20, 37, 70, 135, \ldots$	basis element!
$s(n+3)$:	$11, 20, 37, 70, 135, 264, \ldots$	$= 2s(n) - 5s(n+1) + 4s(n+2)$

Matrix form:

$$\begin{bmatrix} s(n+1) \\ s(n+2) \\ s(n+3) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} s(n) \\ s(n+1) \\ s(n+2) \end{bmatrix}$$

Guessing a 2-regular sequence

$s(n) = \left|\left\{m : \nu_2(\tbinom{n}{m}) = 1\right\}\right|$

$s(n)$:	$0, 0, 1, 0, 1, 2, 2, 0, \ldots$	basis element!
$s(2n+0)$:	$0, 1, 1, 2, 1, 4, 2, 4, \ldots$	basis element!
$s(2n+1)$:	$0, 0, 2, 0, 2, 4, 4, 0, \ldots$	$= 2s(n)$
$s(4n+0)$:	$0, 1, 1, 2, 1, 4, 2, 4, \ldots$	$= s(2n)$
$s(4n+2)$:	$1, 2, 4, 4, 4, 8, 8, 8, \ldots$	basis element!
$s(8n+2)$:	$1, 4, 4, 8, 4, 12, 8, 16, \ldots$	$= -2s(n) + 2s(2n) + s(4n+2)$
$s(8n+6)$:	$2, 4, 8, 8, 8, 16, 16, 16, \ldots$	$= 2s(4n+2)$

Matrix form:

$$\begin{bmatrix} s(2n) \\ s(4n) \\ s(8n+2) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix} = M(0) \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix}$$

$$\begin{bmatrix} s(2n+1) \\ s(4n+2) \\ s(8n+6) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix} = M(1) \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix}$$

An implementation in *Mathematica*

IntegerSequences is available from

`https://people.hofstra.edu/Eric_Rowland/packages.html`

```
In[1]:= Import["https://people.hofstra.edu/Eric_Rowland/packages/IntegerSequences.m"]

In[2]:= Table[Count[Table[IntegerExponent[Binomial[n, m], 2], {m, 0, n}], 1], {n, 0, 31}]

Out[2]= {0, 0, 1, 0, 1, 2, 2, 0, 1, 2, 4, 4, 2, 4, 4, 0, 1, 2, 4, 4, 4, 8, 8, 8, 2, 4, 8, 8, 4, 8, 8, 0}

In[3]:= FindRegularSequenceFunction[%, 2] // RegularSequenceMatrixForm
```

Out[3]= RegularSequence$\left[\{1, 0, 0\}, \left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}\right\}, \{0, 0, 1\}\right]$

Write $n = n_\ell \cdots n_1 n_0$ in base 2; then

$$s(n) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} M(n_0)\, M(n_1) \cdots M(n_\ell) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ (conjecturally).}$$

The matrices $M(0)$ and $M(1)$ aren't unique.
Try many changes of bases. Positive entries permit a bijective proof.

Multinomial coefficients

For a k-tuple $\mathbf{m} = (m_1, m_2, \dots, m_k)$, define

$$\operatorname{total} \mathbf{m} := m_1 + m_2 + \cdots + m_k$$

and

$$\operatorname{mult} \mathbf{m} := \frac{(\operatorname{total} \mathbf{m})!}{m_1!\, m_2! \cdots m_k!}.$$

Theorem (Rowland 2018)

Let $k \geq 1$, and let $e = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{Z}^k$. Write $n = n_\ell \cdots n_1 n_0$ in base p. Then

$$\sum_{\substack{\mathbf{m} \in \mathbb{N}^k \\ \operatorname{total} \mathbf{m} = n}} x^{\nu_p(\operatorname{mult} \mathbf{m})} = e\, M_{p,k}(n_0)\, M_{p,k}(n_1) \cdots M_{p,k}(n_\ell)\, e^\top.$$

$M_{p,k}(d)$ is a $k \times k$ matrix ...

Multinomial coefficients

Example

Let $p = 5$ and $k = 3$; the matrices $M_{5,3}(0), \ldots, M_{5,3}(4)$ are

$$\begin{bmatrix} 1 & 18 & 6 \\ 0 & 15x & 10x \\ 0 & 10x^2 & 15x^2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 19 & 3 \\ x & 18x & 6x \\ 0 & 15x^2 & 10x^2 \end{bmatrix}, \quad \begin{bmatrix} 6 & 18 & 1 \\ 3x & 19x & 3x \\ x^2 & 18x^2 & 6x^2 \end{bmatrix},$$
$$\begin{bmatrix} 10 & 15 & 0 \\ 6x & 18x & x \\ 3x^2 & 19x^2 & 3x^2 \end{bmatrix}, \quad \begin{bmatrix} 15 & 10 & 0 \\ 10x & 15x & 0 \\ 6x^2 & 18x^2 & x^2 \end{bmatrix}.$$

Let $c_{p,k}(n) = \left|\left\{\mathbf{d} \in \{0, \ldots, p-1\}^k : \text{total } \mathbf{d} = n\right\}\right|$. $\quad p = 5$:

$k = 0$:							1						
$k = 1$:					1	1	1	1	1				
$k = 2$:			1	2	3	4	5	4	3	2	1		
$k = 3$:	1	3	6	10	15	18	19	18	15	10	6	3	1

$M_{p,k}(d)$ is the $k \times k$ matrix with entries $c_{p,k}(p(j-1) + d - (i-1))\, x^{i-1}$.

Sketch of proof

Kummer's theorem for multinomial coefficients

Let p be a prime, and let $\mathbf{m} \in \mathbb{N}^k$ for some $k \geq 0$. Then

$$\nu_p(\operatorname{mult} \mathbf{m}) = \frac{\operatorname{total} \sigma_p(\mathbf{m}) - \sigma_p(\operatorname{total} \mathbf{m})}{p-1}.$$

Lemma

Let $k \geq 1$.
Let $0 \leq i \leq k-1$.
Let $d \in \{0, \dots, p-1\}$.
Let $n \geq 0$.
Let $\mathbf{m} \in \mathbb{N}^k$ with $\operatorname{total} \mathbf{m} = pn + d - i$.
Define $j = n - \operatorname{total} \lfloor \mathbf{m}/p \rfloor$.
Then

$$\nu_p(\operatorname{mult} \mathbf{m}) + \nu_p\left(\frac{(pn+d)!}{(pn+d-i)!}\right) = \nu_p(\operatorname{mult} \lfloor \mathbf{m}/p \rfloor) + \nu_p\left(\frac{n!}{(n-j)!}\right) + j.$$

Sketch of proof

Fix $0 \leq i \leq k-1$, $d \in \{0, \dots, p-1\}$, and $\alpha \geq 0$. The map

$$\beta(\mathbf{m}) := (\lfloor \mathbf{m}/p \rfloor, \mathbf{m} \bmod p)$$

is a bijection from

$$A = \left\{ \mathbf{m} \in \mathbb{N}^k : \text{total}\, \mathbf{m} = pn + d - i \text{ and } \nu_p(\text{mult}\, \mathbf{m}) + \nu_p\left(\frac{(pn+d)!}{(pn+d-i)!}\right) = \alpha \right\}$$

to the set

$$B = \bigcup_{j=0}^{k-1} \left(\left\{ \mathbf{c} \in \mathbb{N}^k : \text{total}\, \mathbf{c} = n - j \text{ and } \nu_p(\text{mult}\, \mathbf{c}) + \nu_p\left(\frac{n!}{(n-j)!}\right) + j = \alpha \right\} \times \left\{ \mathbf{d} \in \{0, \dots, p-1\}^k : \text{total}\, \mathbf{d} = pj + d - i \right\} \right).$$

The lemma implies that if $\mathbf{m} \in A$ then $\beta(\mathbf{m}) \in B$.

Bijection example

$k = 2$, $p = 2$, $d = 0$, $i = 0$, $n = 4$, $\alpha = 3$

$$A = \{\mathbf{m} \in \mathbb{N}^2 : \operatorname{total} \mathbf{m} = 8 \text{ and } \nu_2(\operatorname{mult} \mathbf{m}) = 3\}$$
$$= \{(1,7), (3,5), (5,3), (7,1)\}$$

$$\operatorname{mult}(1,7) = 8 \qquad (1,7) = 2\,(0,3) + (1,1)$$
$$\operatorname{mult}(3,5) = 56 \qquad (3,5) = 2\,(1,2) + (1,1)$$

$j = 0$:

$$\{\mathbf{c} \in \mathbb{N}^2 : \operatorname{total} \mathbf{c} = 4 \text{ and } \nu_2(\operatorname{mult} \mathbf{c}) = 3\} \times \{\mathbf{d} \in \{0,1\}^2 : \operatorname{total} \mathbf{d} = 0\}$$
$$= \{\} \times \{(0,0)\}$$

$j = 1$:

$$\{\mathbf{c} \in \mathbb{N}^2 : \operatorname{total} \mathbf{c} = 3 \text{ and } \nu_2(\operatorname{mult} \mathbf{c}) = 0\} \times \{\mathbf{d} \in \{0,1\}^2 : \operatorname{total} \mathbf{d} = 2\}$$
$$= \{(0,3), (1,2), (2,1), (3,0)\} \times \{(1,1)\}$$

Unexplored territory

Do generalizations of binomial coefficients have analogous products?

- Fibonomial coefficients
- *q*-binomial coefficients
- Carlitz binomial coefficients
- other hypergeometric terms $\qquad \binom{n}{m} = \frac{n!}{m!(n-m)!}$
- coefficients in other rational series $\qquad \binom{n+m}{m} = [x^n y^m] \frac{1}{1-x-y}$