Enumeration of binomial coefficients by their p-adic valuations

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Valuations of binomial coefficients

Pascal's triangle:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

For this talk: p is a prime.

Let $\nu_p(n)$ denote the exponent of the highest power of p dividing n.
Example: $\nu_3(18) = 2$.

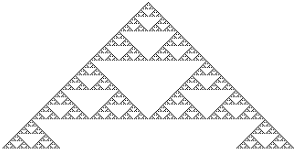
Theorem (Kummer 1852)

$\nu_p(\binom{n}{m}) =$ *# carries involved in adding* m *to* $n - m$ *in base* p.

Odd binomial coefficients

Main theme: Arithmetic information about $\binom{n}{m}$ reflects the base-p representations of n, m.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$



Glaisher (1899) counted odd binomial coefficients:

$$1, 2, 2, 4, 2, 4, 4, 8, 2, 4, 4, 8, 4, 8, 8, 16, \ldots \qquad 2^{|n|_1} = \theta_{2,0}(n)$$

$|n|_d$:= # occurrences of d in the base-p representation of n

$\theta_{p,\alpha}(n) := \left|\{m : 0 \leq m \leq n \text{ and } \nu_p(\binom{n}{m}) = \alpha\}\right|$

Derivation from Kummer's theorem

Glaisher's result $\theta_{2,0}(n) = 2^{|n|_1}$ follows from Kummer's theorem.

Theorem (Kummer)

$\nu_p(\binom{n}{m})$ = *# carries involved in adding m to $n - m$ in base p.*

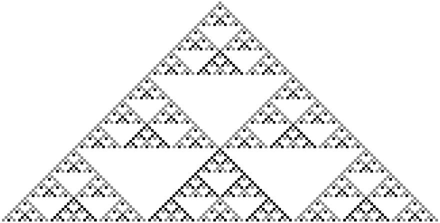
Example

$n = 25$. How many m satisfy $\nu_2(\binom{25}{m}) = 0$?

$$n = 25 = 11001_2$$
$$m = **00*_2$$

$\theta_{2,0}(25) = 2^{|25|_1} = 8$.

Binomial coefficients not divisible by p



Number of binomial coefficients with 3-adic valuation 0:

$$1, 2, 3, 2, 4, 6, 3, 6, 9, 2, 4, 6, 4, 8, 12, 6, \dots \qquad \theta_{3,0}(n) = 2^{|n|_1} 3^{|n|_2}$$

Theorem (Fine 1947)

Write $n = n_\ell \cdots n_1 n_0$ in base p. Then

$$\theta_{p,0}(n) = (n_0 + 1) \cdots (n_\ell + 1) = 1^{|n|_0} 2^{|n|_1} 3^{|n|_2} \cdots p^{|n|_{p-1}}.$$

Prime powers?

Carlitz found a recurrence for $\theta_{p,\alpha}(n)$ involving

$$\psi_{p,\alpha}(n) := \left|\{m \,:\, 0 \leq m \leq n \text{ and } \nu_p((m+1)\tbinom{n}{m}) = \alpha\}\right|.$$

Theorem (Carlitz 1967)

$$\theta_{p,\alpha}(pn+d) = (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1)$$

$$\psi_{p,\alpha}(pn+d) = \begin{cases} (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1) & \text{if } 0 \leq d \leq p-2 \\ p\psi_{p,\alpha-1}(n) & \text{if } d = p-1. \end{cases}$$

Is there a better formulation of this recurrence?

Recurrences with constant coefficients

Constant-recursive sequences

Fibonacci recurrence: $F(n+2) = F(n+1) + F(n)$

Matrix form: $$\begin{bmatrix} F(n+1) \\ F(n+2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F(n) \\ F(n+1) \end{bmatrix}$$

Matrix product: $$F(n) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Characterizations of constant-recursive sequences over $\mathbb{Q}$:

- $s(n)$ is determined by a linear recurrence in $s(n+i)$ (along with finitely many initial conditions)
- $\langle \{ s(n+i)_{n \geq 0} : i \geq 0 \} \rangle$ is finite-dimensional
- $s(n) = u\, M^n\, v$ for some matrix M and vectors u, v
- generating function $\sum_{n \geq 0} s(n) x^n$ is rational

k-regularity

Definition

Let $k \geq 2$.
A sequence $s(n)_{n \geq 0}$ is k-regular if the vector space generated by

$$\{s(k^e n + i)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq i \leq k^e - 1\}$$

is finite-dimensional.

Characterizations of k-regularity:

- $s(n)$ is determined by finitely many linear recurrences in $s(k^e n + i)$ (along with finitely many initial conditions)
- $s(n) = u\, M(n_0)\, M(n_1) \cdots M(n_\ell)\, v$ for some $M(d)$ and vectors u, v
- generating function in k non-commuting variables is rational

How to guess a recurrence?

Guessing a constant-recursive sequence

$\langle\{s(n+i)_{n\geq 0} : i \geq 0\}\rangle$ is finite-dimensional.

$s(n) = 2^n + n$:

$$\begin{array}{rll}
s(n): & 1, 3, 6, 11, 20, 37, \ldots & \text{basis element!} \\
s(n+1): & 3, 6, 11, 20, 37, 70, \ldots & \text{basis element!} \\
s(n+2): & 6, 11, 20, 37, 70, 135, \ldots & \text{basis element!} \\
s(n+3): & 11, 20, 37, 70, 135, 264, \ldots & = 2s(n) - 5s(n+1) + 4s(n+2)
\end{array}$$

Matrix form:

$$\begin{bmatrix} s(n+1) \\ s(n+2) \\ s(n+3) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} s(n) \\ s(n+1) \\ s(n+2) \end{bmatrix}$$

Guessing a 2-regular sequence

$s(n) = \theta_{2,1}(n) =$ # binomial coefficients $\binom{n}{m}$ with $\nu_2(\binom{n}{m}) = 1$:

$s(n)$:	$0, 0, 1, 0, 1, 2, 2, 0, \dots$	basis element!
$s(2n+0)$:	$0, 1, 1, 2, 1, 4, 2, 4, \dots$	basis element!
$s(2n+1)$:	$0, 0, 2, 0, 2, 4, 4, 0, \dots$	$= 2s(n)$
$s(4n+0)$:	$0, 1, 1, 2, 1, 4, 2, 4, \dots$	$= s(2n)$
$s(4n+2)$:	$1, 2, 4, 4, 4, 8, 8, 8, \dots$	basis element!
$s(8n+2)$:	$1, 4, 4, 8, 4, 12, 8, 16, \dots$	$= -2s(n) + 2s(2n) + s(4n+2)$
$s(8n+6)$:	$2, 4, 8, 8, 8, 16, 16, 16, \dots$	$= 2s(4n+2)$

Matrix form:

$$\begin{bmatrix} s(2n) \\ s(4n) \\ s(8n+2) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix} = M(0) \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix}$$

$$\begin{bmatrix} s(2n+1) \\ s(4n+2) \\ s(8n+6) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix} = M(1) \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix}$$

An implementation in *Mathematica*

IntegerSequences is available from

https://people.hofstra.edu/Eric_Rowland/packages.html

```
In[1]:= Import["https://people.hofstra.edu/Eric_Rowland/packages/IntegerSequences.m"]

In[2]:= Table[Count[Table[IntegerExponent[Binomial[n, m], 2], {m, 0, n}], 1], {n, 0, 31}]

Out[2]= {0, 0, 1, 0, 1, 2, 2, 0, 1, 2, 4, 4, 2, 4, 4, 0, 1, 2, 4, 4, 4, 8, 8, 8, 2, 4, 8, 8, 4, 8, 8, 0}

In[3]:= FindRegularSequenceFunction[%, 2] // RegularSequenceMatrixForm
```

Out[3]= $\text{RegularSequence}\left[\{1, 0, 0\}, \left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}\right\}, \{0, 0, 1\}\right]$

Sequences of polynomials

Fibonacci numbers

$$\color{red}{F(n)} = \#\text{ compositions of } n-1 \text{ using } 1, 2$$

$n = 5$: $\quad 1+1+1+1 \quad 1+1+2 \quad 1+2+1 \quad 2+1+1 \quad 2+2$

n	$F(n)$
0	0
1	1
2	1
3	2
4	3
5	5
6	8
7	13

Fibonacci polynomials

Refinement:

$$\color{red}{F(n,x)} := \sum_{\substack{\text{compositions } \lambda \text{ of} \\ n-1 \text{ using } 1,2}} x^{|\lambda|_1}$$

The coefficient of x^α is the number of compositions with α 1s.

$n = 5$: $1+1+1+1$ $1+1+2$ $1+2+1$ $2+1+1$ $2+2$

n	$F(n)$	$F(n,x)$
0	0	0
1	1	1
2	1	x
3	2	$x^2 + 1$
4	3	$x^3 + 2x$
5	5	$1x^4 + 3x^2 + 1$
6	8	$x^5 + 4x^3 + 3x$
7	13	$x^6 + 5x^4 + 6x^2 + 1$

Fibonacci polynomials

Recurrence:

$$F(n+2,x) = x\,F(n+1,x) + F(n,x)$$

Matrix form:

$$\begin{bmatrix} F(n+1,x) \\ F(n+2,x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & x \end{bmatrix} \begin{bmatrix} F(n,x) \\ F(n+1,x) \end{bmatrix}$$

Matrix product:

$$F(n,x) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & x \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Generating function

$$T_p(n, x) := \sum_{m=0}^{n} x^{\nu_p(\binom{n}{m})}$$

The coefficient of x^α is the number of $\binom{n}{m}$ with p-adic valuation α.

$n = 8$:	1	8	28	56	70	56	28	8	1
$\nu_2(\binom{8}{m})$:	0	3	2	3	1	3	2	3	0

n	$T_2(n, x)$
0	1
1	2
2	$x + 2$
3	4
4	$2x^2 + x + 2$
5	$2x + 4$
6	$x^2 + 2x + 4$
7	8
8	$4x^3 + 2x^2 + 1x + 2$

Guessing matrices for $T_p(n, x)$

$p = 2$:

$$M_2(0) = \begin{bmatrix} 0 & 1 \\ -2x & 2x+1 \end{bmatrix} \quad M_2(1) = \begin{bmatrix} 2 & 0 \\ 2 & x \end{bmatrix}$$

$p = 3$:

$$M_3(0) = \begin{bmatrix} 0 & 1 \\ -3x & 3x+1 \end{bmatrix} \quad M_3(1) = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & 2x + \frac{1}{2} \end{bmatrix} \quad M_3(2) = \begin{bmatrix} 3 & 0 \\ 3x+3 & x \end{bmatrix}$$

$p = 5$:

$$M_5(0) = \begin{bmatrix} 0 & 1 \\ -5x & 5x+1 \end{bmatrix} \quad M_5(1) = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{5}{4} & 4x + \frac{3}{4} \end{bmatrix} \quad \cdots \quad M_5(4) = \begin{bmatrix} 5 & 0 \\ 15x+5 & x \end{bmatrix}$$

General p:

$$M_p(d) = \begin{bmatrix} \frac{dp}{p-1} & \frac{p-1-d}{p-1} \\ (d-1)px + \frac{dp}{p-1} & (p-d)x + \frac{p-1-d}{p-1} \end{bmatrix}$$

$$(0 \leq d \leq p-1)$$

But this matrix is not unique. There are many bases.

Which basis is best?

Can we get integer coefficients?
Can we get non-negative coefficients? (toward a bijective proof)

Try all 2×2 invertible change-of-basis matrices S with entries $\leq j$:

$$S^{-1} M_p(d) S.$$

$$T_p(n, x) = \begin{bmatrix} 1 & 0 \end{bmatrix} M_p(n_0)\, M_p(n_1) \cdots M_p(n_\ell) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Maximize the number of monomial entries:

$$\begin{bmatrix} d+1 & p-d-1 \\ d\,x & (p-d)\,x \end{bmatrix}$$

Matrix product

$$M_p(d) := \begin{bmatrix} d+1 & p-d-1 \\ d\,x & (p-d)\,x \end{bmatrix}$$

Theorem (Rowland 2018)

Write $n = n_\ell \cdots n_1 n_0$ in base p. Then

$$T_p(n,x) := \sum_{m=0}^{n} x^{\nu_p(\binom{n}{m})} = \begin{bmatrix} 1 & 0 \end{bmatrix} M_p(n_0)\, M_p(n_1) \, \cdots \, M_p(n_\ell) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Example

$p = 2$, $n = 8 = 1000_2$:

$$\begin{aligned} T_2(8,x) &= \begin{bmatrix} 1 & 0 \end{bmatrix} M_2(0) M_2(0) M_2(0) M_2(1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2\,x \end{bmatrix}^3 \begin{bmatrix} 2 & 0 \\ x & x \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= 4x^3 + 2x^2 + x + 2. \end{aligned}$$

Comparing recurrences

Carlitz recurrence:

$$\theta_{p,\alpha}(pn+d) = (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1)$$

$$\psi_{p,\alpha}(pn+d) = \begin{cases} (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1) & \text{if } 0 \le d \le p-2 \\ p\psi_{p,\alpha-1}(n) & \text{if } d = p-1. \end{cases}$$

Carlitz has $\psi_{p,\alpha}(pn+d)$ on the left but $\psi_{p,\alpha-1}(n-1)$ on the right.

Recurrence leading to matrix product:

$$\theta_{p,\alpha}(pn+d) = (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1)$$
$$\psi_{p,\alpha}(pn+d-1) = d\theta_{p,\alpha}(n) + (p-d)\psi_{p,\alpha-1}(n-1).$$

$$M_p(d) = \begin{bmatrix} d+1 & p-d-1 \\ d\,x & (p-d)\,x \end{bmatrix}$$

Multinomial coefficients

For a k-tuple $\mathbf{m} = (m_1, m_2, \ldots, m_k)$ of non-negative integers, define

$$\text{total}\,\mathbf{m} := m_1 + m_2 + \cdots + m_k$$

and

$$\text{mult}\,\mathbf{m} := \frac{(\text{total}\,\mathbf{m})!}{m_1!\,m_2!\,\cdots\,m_k!}.$$

Theorem (Rowland 2018)

Let $k \geq 1$, and let $e = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{Z}^k$.
Write $n = n_\ell \cdots n_1 n_0$ in base p. Then

$$\sum_{\substack{\mathbf{m} \in \mathbb{N}^k \\ \text{total}\,\mathbf{m} = n}} x^{\nu_p(\text{mult}\,\mathbf{m})} = e\, M_{p,k}(n_0)\, M_{p,k}(n_1) \cdots M_{p,k}(n_\ell)\, e^\top.$$

$M_{p,k}(d)$ is a $k \times k$ matrix …

Multinomial coefficients

Example

Let $p = 5$ and $k = 3$; the matrices $M_{5,3}(0), \ldots, M_{5,3}(4)$ are

$$\begin{bmatrix} 1 & 18 & 6 \\ 0 & 15x & 10x \\ 0 & 10x^2 & 15x^2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 19 & 3 \\ x & 18x & 6x \\ 0 & 15x^2 & 10x^2 \end{bmatrix}, \quad \begin{bmatrix} 6 & 18 & 1 \\ 3x & 19x & 3x \\ x^2 & 18x^2 & 6x^2 \end{bmatrix},$$
$$\begin{bmatrix} 10 & 15 & 0 \\ 6x & 18x & x \\ 3x^2 & 19x^2 & 3x^2 \end{bmatrix}, \quad \begin{bmatrix} 15 & 10 & 0 \\ 10x & 15x & 0 \\ 6x^2 & 18x^2 & x^2 \end{bmatrix}.$$

Let $c_{p,k}(n) = \left|\{\mathbf{d} \in \{0, \ldots, p-1\}^k : \text{total}\, \mathbf{d} = n\}\right|$. $p = 5$:

$k = 0$:							1						
$k = 1$:					1	1	1	1	1				
$k = 2$:			1	2	3	4	5	4	3	2	1		
$k = 3$:	1	3	6	10	15	18	19	18	15	10	6	3	1

$M_{p,k}(d)$ is the $k \times k$ matrix with entries $c_{p,k}(p\,(j-1) + d - (i-1))\, x^{i-1}$.

Sketch of proof

Lemma

Let $k \geq 1$*.*
Let $0 \leq i \leq k - 1$*.*
Let $d \in \{0, \dots, p-1\}$*.*
Let $n \geq 0$*.*
Let $\mathbf{m} \in \mathbb{N}^k$ *with* $\operatorname{total} \mathbf{m} = pn + d - i$*.*
Define $j = n - \operatorname{total} \lfloor \mathbf{m}/p \rfloor$*.*
Then

$$\nu_p(\operatorname{mult} \mathbf{m}) + \nu_p\left(\frac{(pn+d)!}{(pn+d-i)!}\right) = \nu_p(\operatorname{mult} \lfloor \mathbf{m}/p \rfloor) + \nu_p\left(\frac{n!}{(n-j)!}\right) + j.$$

Sketch of proof

For $d \in \{0, \dots, p-1\}$, $0 \le i \le k-1$, and $\alpha \ge 0$, the map

$$\beta(\mathbf{m}) := (\lfloor \mathbf{m}/p \rfloor, \mathbf{m} \bmod p)$$

is a bijection from the set

$$A = \left\{ \mathbf{m} \in \mathbb{N}^k : \text{total}\, \mathbf{m} = pn + d - i \text{ and } \nu_p(\text{mult}\, \mathbf{m}) + \nu_p\left(\frac{(pn+d)!}{(pn+d-i)!} \right) = \alpha \right\}$$

to the set

$$B = \bigcup_{j=0}^{k-1} \left(\left\{ \mathbf{c} \in \mathbb{N}^k : \text{total}\, \mathbf{c} = n - j \text{ and } \nu_p(\text{mult}\, \mathbf{c}) + \nu_p\left(\frac{n!}{(n-j)!} \right) + j = \alpha \right\} \times \left\{ \mathbf{d} \in \{0, \dots, p-1\}^k : \text{total}\, \mathbf{d} = pj + d - i \right\} \right).$$

The lemma implies that if $\mathbf{m} \in A$ then $\beta(\mathbf{m}) \in B$.

Bijection example

$k = 2$, $p = 2$, $d = 0$, $i = 0$, $n = 4$, $\alpha = 3$

$$A = \{\mathbf{m} \in \mathbb{N}^2 : \operatorname{total} \mathbf{m} = 8 \text{ and } \nu_2(\operatorname{mult} \mathbf{m}) = 3\}$$
$$= \{(1,7), (3,5), (5,3), (7,1)\}$$

$$\operatorname{mult}(1,7) = 8 \qquad (1,7) = 2\,(0,3) + (1,1)$$
$$\operatorname{mult}(3,5) = 56 \qquad (3,5) = 2\,(1,2) + (1,1)$$

$j = 0$:

$$\{\mathbf{c} \in \mathbb{N}^2 : \operatorname{total} \mathbf{c} = 4 \text{ and } \nu_2(\operatorname{mult} \mathbf{c}) = 3\} \times \{\mathbf{d} \in \{0,1\}^2 : \operatorname{total} \mathbf{d} = 0\}$$
$$= \{\} \times \{(0,0)\}$$

$j = 1$:

$$\{\mathbf{c} \in \mathbb{N}^2 : \operatorname{total} \mathbf{c} = 3 \text{ and } \nu_2(\operatorname{mult} \mathbf{c}) = 0\} \times \{\mathbf{d} \in \{0,1\}^2 : \operatorname{total} \mathbf{d} = 2\}$$
$$= \{(0,3), (1,2), (2,1), (3,0)\} \times \{(1,1)\}$$

Unexplored territory

Do generalizations of binomial coefficients have analogous products?

- Fibonomial coefficients
- q-binomial coefficients
- Carlitz binomial coefficients
- other hypergeometric terms $\qquad \binom{n}{m} = \frac{n!}{m!(n-m)!}$
- coefficients in other rational series $\qquad \binom{n+m}{m} = [x^n y^m] \frac{1}{1-x-y}$