# Enumeration of binomial coefficients by their $p$-adic valuations

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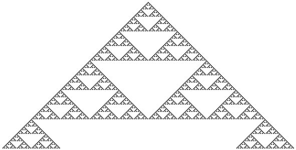
# Odd binomial coefficients

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 1 | | | | |
| | | | 1 | | 1 | | | |
| | | 1 | | 2 | | 1 | | |
| | 1 | | 3 | | 3 | | 1 | |
| 1 | | 4 | | 6 | | 4 | | 1 |



Glaisher (1899): How many odd entries are on each row?

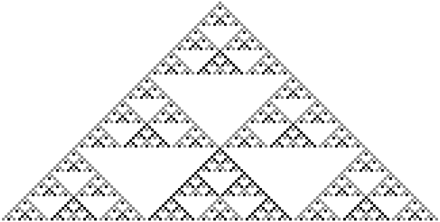
$1, 2, 2, 4, 2, 4, 4, 8, 2, 4, 4, 8, 4, 8, 8, 16, \ldots \qquad 2^{|n|_1}$

$|n|_d :=$ number of occurrences of $d$ in the base-$p$ representation of $n$.

**Main theme**

*Arithmetic information about $\binom{n}{m}$ reflects base-p representations.*

# Fine's theorem



Number of binomial coefficients not divisible by 3:

$1, 2, 3, 2, 4, 6, 3, 6, 9, 2, 4, 6, 4, 8, 12, 6, \ldots \qquad 2^{|n|_1} 3^{|n|_2}$

**Theorem (Fine 1947)**

*Write $n = n_\ell \cdots n_1 n_0$ in base $p$. Then*

$$\left|\left\{m : \binom{n}{m} \text{ is not divisible by } p\right\}\right| = (n_0 + 1)(n_1 + 1) \cdots (n_\ell + 1)$$
$$= 1^{|n|_0} 2^{|n|_1} 3^{|n|_2} \cdots p^{|n|_{p-1}}.$$

# Prime powers?

$\nu_p(n) := \max\{e \geq 0 : p^e \text{ divides } n\}$.
Example: $\nu_3(18) = 2$.

Carlitz found a recurrence involving
$\theta_{p,\alpha}(n) := \left|\{m : 0 \leq m \leq n \text{ and } \nu_p(\binom{n}{m}) = \alpha\}\right|$ and
$\psi_{p,\alpha}(n) := \left|\{m : 0 \leq m \leq n \text{ and } \nu_p((m+1)\binom{n}{m}) = \alpha\}\right|$.

**Theorem (Carlitz 1967)**

$$\theta_{p,\alpha}(pn+d) = (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1)$$

$$\psi_{p,\alpha}(pn+d) = \begin{cases} (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1) & \text{if } 0 \leq d \leq p-2 \\ p\psi_{p,\alpha-1}(n) & \text{if } d = p-1. \end{cases}$$

Is there a better recurrence/formulation?

# Generating function

Define

$$\textcolor{red}{T_p(n, x)} := \sum_{m=0}^{n} x^{\nu_p(\binom{n}{m})} = \sum_{\alpha \geq 0} \theta_{p,\alpha}(n) x^\alpha.$$

$p = 2$:

| $n$ | | | | | | | | | | $T_2(n,x)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | 1 | | | | | $1$ |
| 1 | | | | 1 | | 1 | | | | $2$ |
| 2 | | | 1 | | 2 | | 1 | | | $x+2$ |
| 3 | | 1 | | 3 | | 3 | | 1 | | $4$ |
| 4 | 1 | | 4 | | 6 | | 4 | | 1 | $2x^2 + x + 2$ |
| 5 | | | | | | | | | | $2x + 4$ |
| 6 | | | | | $\vdots$ | | | | | $x^2 + 2x + 4$ |
| 7 | | | | | | | | | | $8$ |

In particular, $T_p(n, 1) = n + 1$.

# $k$-regularity

## Definition (Allouche–Shallit 1992)

Let $k \geq 2$.
A sequence $s(n)_{n\geq 0}$ is $k$-regular if the vector space generated by

$$\{s(k^e n + i)_{n\geq 0} : e \geq 0 \text{ and } 0 \leq i \leq k^e - 1\}$$

is finite-dimensional.

Compare to:

## Definition

$s(n)_{n\geq 0}$ is constant-recursive if the vector space generated by $\{s(n+i)_{n\geq 0} : i \geq 0\}$ is finite-dimensional. Equivalently:

- $s(n)$ satisfies a linear recurrence involving $s(n+i)$
- $s(n) = u\, M^n\, v$ for some matrix $M$ and vectors $u, v$
- the generating function $\sum_{n\geq 0} s(n) x^n$ is rational

# Guessing a 2-regular sequence

$s(n) = \left|\left\{m : \nu_2(\binom{n}{m}) = 1\right\}\right|$

| | | |
|---|---|---|
| $s(n)$ : | $0, 0, 1, 0, 1, 2, 2, 0, \ldots$ | basis element! |
| $s(2n+0)$ : | $0, 1, 1, 2, 1, 4, 2, 4, \ldots$ | basis element! |
| $s(2n+1)$ : | $0, 0, 2, 0, 2, 4, 4, 0, \ldots$ | $= 2s(n)$ |
| $s(4n+0)$ : | $0, 1, 1, 2, 1, 4, 2, 4, \ldots$ | $= s(2n)$ |
| $s(4n+2)$ : | $1, 2, 4, 4, 4, 8, 8, 8, \ldots$ | basis element! |
| $s(8n+2)$ : | $1, 4, 4, 8, 4, 12, 8, 16, \ldots$ | $= -2s(n) + 2s(2n) + s(4n+2)$ |
| $s(8n+6)$ : | $2, 4, 8, 8, 8, 16, 16, 16, \ldots$ | $= 2s(4n+2)$ |

Matrix form:

$$\begin{bmatrix} s(2n) \\ s(4n) \\ s(8n+2) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix} = M(0) \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix}$$

$$\begin{bmatrix} s(2n+1) \\ s(4n+2) \\ s(8n+6) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix} = M(1) \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix}$$

# An implementation in *Mathematica*

Write $n = n_\ell \cdots n_1 n_0$ in base 2; then

$$s(n) = \begin{bmatrix}1 & 0 & 0\end{bmatrix} M(n_0)\, M(n_1)\, \cdots\, M(n_\ell) \begin{bmatrix}0\\0\\1\end{bmatrix}$$ (conjecturally).

**IntegerSequences** is available from

`https://people.hofstra.edu/Eric_Rowland/packages.html`

```
In[1]:= Import["https://people.hofstra.edu/Eric_Rowland/packages/IntegerSequences.m"]

In[2]:= Table[Count[Table[IntegerExponent[Binomial[n, m], 2], {m, 0, n}], 1], {n, 0, 31}]
Out[2]= {0, 0, 1, 0, 1, 2, 2, 0, 1, 2, 4, 4, 2, 4, 4, 0, 1, 2, 4, 4, 4, 8, 8, 8, 2, 4, 8, 8, 4, 8, 8, 0}

In[3]:= FindRegularSequenceFunction[%, 2] // RegularSequenceMatrixForm
```

Out[3]= $\text{RegularSequence}\left[\{1, 0, 0\}, \left\{\begin{pmatrix}0 & 1 & 0\\ 0 & 1 & 0\\ -2 & 2 & 1\end{pmatrix}, \begin{pmatrix}2 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 2\end{pmatrix}\right\}, \{0, 0, 1\}\right]$

The matrices $M(0)$ and $M(1)$ aren't unique. (There are many bases.)
Positive entries permit bijective proofs.

# Matrix generalization of Fine's theorem

Let

$$M_p(d) := \begin{bmatrix} d+1 & p-d-1 \\ d\,x & (p-d)\,x \end{bmatrix}.$$

## Theorem (Rowland 2018)

*Write $n = n_\ell \cdots n_1 n_0$ in base $p$. Then*

$$T_p(n,x) := \sum_{m=0}^{n} x^{\nu_p(\binom{n}{m})} = \begin{bmatrix} 1 & 0 \end{bmatrix} M_p(n_0)\, M_p(n_1) \cdots M_p(n_\ell) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

## Example

$p = 2, \quad n = 6 = 110_2$:

$$T_2(6,x) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2x \end{bmatrix} \begin{bmatrix} 2 & 0 \\ x & x \end{bmatrix} \begin{bmatrix} 2 & 0 \\ x & x \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x^2 + 2x + 4.$$

# Multinomial coefficients

For a $k$-tuple $\mathbf{m} = (m_1, m_2, \dots, m_k)$ of non-negative integers, define

$$\operatorname{total} \mathbf{m} := m_1 + m_2 + \cdots + m_k$$

and

$$\operatorname{mult} \mathbf{m} := \frac{(\operatorname{total} \mathbf{m})!}{m_1!\, m_2! \, \cdots \, m_k!}.$$

## Theorem (Rowland 2018)

*Let $k \geq 1$, and let $e = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{Z}^k$. Write $n = n_\ell \cdots n_1 n_0$ in base $p$. Then*

$$\sum_{\substack{\mathbf{m} \in \mathbb{N}^k \\ \operatorname{total} \mathbf{m} = n}} x^{\nu_p(\operatorname{mult} \mathbf{m})} = e\, M_{p,k}(n_0)\, M_{p,k}(n_1) \cdots M_{p,k}(n_\ell)\, e^\top.$$

$M_{p,k}(d)$ is a $k \times k$ matrix ...

# Multinomial coefficients

## Example

Let $p = 5$ and $k = 3$; the matrices $M_{5,3}(0), \dots, M_{5,3}(4)$ are

$$\begin{bmatrix} 1 & 18 & 6 \\ 0 & 15x & 10x \\ 0 & 10x^2 & 15x^2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 19 & 3 \\ x & 18x & 6x \\ 0 & 15x^2 & 10x^2 \end{bmatrix}, \quad \begin{bmatrix} 6 & 18 & 1 \\ 3x & 19x & 3x \\ x^2 & 18x^2 & 6x^2 \end{bmatrix},$$

$$\begin{bmatrix} 10 & 15 & 0 \\ 6x & 18x & x \\ 3x^2 & 19x^2 & 3x^2 \end{bmatrix}, \quad \begin{bmatrix} 15 & 10 & 0 \\ 10x & 15x & 0 \\ 6x^2 & 18x^2 & x^2 \end{bmatrix}.$$

Let $c_{p,k}(n) = \left|\left\{\mathbf{d} \in \{0, \dots, p-1\}^k : \text{total}\, \mathbf{d} = n\right\}\right|$. $\quad p = 5$:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k = 0$: | | | | | | | 1 | | | | | | |
| $k = 1$: | | | | | 1 | 1 | 1 | 1 | 1 | | | | |
| $k = 2$: | | | 1 | 2 | 3 | 4 | 5 | 4 | 3 | 2 | 1 | | |
| $k = 3$: | 1 | 3 | 6 | 10 | 15 | 18 | 19 | 18 | 15 | 10 | 6 | 3 | 1 |

$M_{p,k}(d)$ is the $k \times k$ matrix with entries $c_{p,k}(p\,(j-1) + d - (i-1))\, x^{i-1}$.

# Lemma with many variables

## Lemma

*Let* $n \geq 0$*.*
*Let* $k \geq 1$*.*
*Let* $0 \leq i \leq k - 1$*.*
*Let* $d \in \{0, \ldots, p - 1\}$*.*
*Let* $\mathbf{m} \in \mathbb{N}^k$ *with* $\operatorname{total} \mathbf{m} = pn + d - i$*.*
*Define* $j = n - \operatorname{total}\lfloor \mathbf{m}/p \rfloor$*.*
*Then* $\operatorname{total}(\mathbf{m} \bmod p) = pj + d - i$*,* $0 \leq j \leq k - 1$*, and*

$$\nu_p(\operatorname{mult} \mathbf{m}) + \nu_p\left(\frac{(pn + d)!}{(pn + d - i)!}\right) = \nu_p(\operatorname{mult}\lfloor \mathbf{m}/p \rfloor) + \nu_p\left(\frac{n!}{(n - j)!}\right) + j.$$

# Unexplored territory

Do generalizations of binomial coefficients have analogous products?

- Fibonomial coefficients
- *q*-binomial coefficients
- Carlitz binomial coefficients
- other hypergeometric terms $\qquad \binom{n}{m} = \frac{n!}{m!(n-m)!}$
- coefficients in other rational series $\qquad \binom{n+m}{m} = [x^n y^m]\frac{1}{1-x-y}$