Distinguished equivalence classes of words in F_2

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Outline

Notation

- $L_2 = \{a, b, \overline{a}, \overline{b}\}$, where $\overline{a} = a^{-1}$ and $\overline{b} = b^{-1}$.
- The free group on two generators:
  $F_2 = \langle a, b\rangle = \{w_1 \cdots w_\ell \in L_2^* : w_i \neq w_{i+1}^{-1}$ for $1 \leq i \leq \ell - 1\}$
  $= \{\epsilon, a, b, \overline{a}, \overline{b}, aa, ab, a\overline{b}, ba, bb, b\overline{a}, \overline{a}b, \overline{a}\overline{a}, \overline{a}\overline{b}, \overline{b}a, \overline{b}\overline{a}, \overline{b}\overline{b}, \dots\}$.

Definition

A word $w \in F_2$ is minimal if $|w| \leq |\phi(w)|$ for all $\phi \in \operatorname{Aut} F_2$.

Example

Let $\pi(a) = b, \pi(b) = \overline{a}$.
π extends to an automorphism of F_2: $\pi^{-1}(a) = \overline{b}$ and $\pi^{-1}(b) = a$.

A permutation is an automorphism that permutes L_2.

One-letter automorphisms

Definition

A one-letter automorphism of F_2 is an automorphism which maps $x \mapsto x$ and $y \mapsto yx$ for some $x, y \in L_2$ with $y \notin \{x, \overline{x}\}$.

Example

Let $\phi(a) = a, \phi(b) = ba$.
Then $\phi^{-1}(a) = a, \phi^{-1}(b) = b\overline{a}$.
We have $\phi(a\overline{b}) = a\overline{a}\overline{b} = \overline{b}$, so $a\overline{b}$ is not minimal.

Whitehead's theorem

We write $w \sim v$ if $\phi(w) = v$ for some automorphism ϕ.

Theorem (Whitehead, 1936)

If $w, v \in F_2$ such that $w \sim v$ and v is minimal, then there exists a sequence $\phi_1, \phi_2, \ldots, \phi_m$ of one-letter automorphisms such that

- *$\phi_m \cdots \phi_2 \phi_1(w)$ and v differ only by a permutation and an inner automorphism and*
- *for $0 \leq k \leq m - 1$, $|\phi_{k+1}\phi_k \cdots \phi_2\phi_1(w)| < |\phi_k \cdots \phi_2\phi_1(w)|$.*

Algorithms

The set of one-letter automorphisms is finite.

Corollary

There is an algorithm for determining whether $w \in F_2$ is minimal.
There is an algorithm for determining whether $w, v \in F_2$ are equivalent.

- Myasnikov and Shpilrain (2003): The number of minimal words equivalent to minimal $w \in F_2$ is bounded by a polynomial in $|w|$.
- Lee (2006): The same is true for minimal $w \in F_n$ under a local condition on w.

These results imply upper bounds on the time required to determine whether $w, v \in F_n$ are equivalent.

Computing equivalence classes

Corollary

There is an algorithm, given n, for computing representatives of all equivalence classes of F_2 that contain some word of length $\leq n$.

We computed all classes containing a word of length ≤ 20.

Represent a class W by its lexicographically least minimal word.

Example

$$W = \{aa, bb, \bar{a}\bar{a}, \bar{b}\bar{b}, abab, abb\bar{a}, a\bar{b}a\bar{b}, a\bar{b}\bar{b}\bar{a}, baa\bar{b}, baba, b\bar{a}b\bar{a}, b\bar{a}\bar{a}\bar{b},$$
$$\bar{a}bba, \bar{a}b\bar{a}b, \bar{a}\bar{b}\bar{a}b, \bar{a}\bar{b}\bar{b}a, \bar{b}aab, \bar{b}a\bar{b}a, \bar{b}\bar{a}\bar{a}b, \bar{b}\bar{a}\bar{b}\bar{a}, aabaab, \ldots\}.$$

The minimal words in W are $\{aa, bb, \bar{a}\bar{a}, \bar{b}\bar{b}\}$, and the lex least is aa.

Equivalence classes

0.1	ϵ
1.1	a
2.1	aa
3.1	aaa
4.1	$aaaa$
4.2	$ab\bar{a}\bar{b}$
4.3	$aabb$
5.1	$aaaaa$
5.2	$aaba\bar{b}$
5.3	$aab\bar{a}\bar{b}$
5.4	$aaabb$

6.1	$aaaaaa$
6.2	$aaaba\bar{b}$
6.3	$aaabbb$
6.4	$aaab\bar{a}\bar{b}$
6.5	$aabaa\bar{b}$
6.6	$aaba\overline{bb}$
6.7	$aabba\bar{b}$
6.8	$aabb\bar{a}\bar{b}$
6.9	$aab\overline{aa}\bar{b}$
6.10	$aaaabb$

7.1	$aaaaaaa$
7.2	$aaaaba\bar{b}$
7.3	$aaaabbb$

7.4	$aaaab\bar{a}\bar{b}$
7.5	$aaabaa\bar{b}$
7.6	$aaaba\overline{bb}$
7.7	$aaabba\bar{b}$
7.8	$aaabb\bar{a}b$
7.9	$aaabb\bar{a}\bar{b}$
7.10	$aaab\bar{a}bb$
7.11	$aaab\overline{aa}\bar{b}$
7.12	$aaab\bar{a}\overline{bb}$
7.13	$aabaa\overline{bb}$
7.14	$aabba\overline{bb}$
7.15	$aabb\overline{aa}\bar{b}$
7.16	$aaaaabb$

Outline

Characterization of minimal words

Definition

Let $(v)_w$ denote the number of (possibly overlapping) occurrences of v and v^{-1} in the cyclic word w.

Example

Let $w = aa\bar{b}\bar{b}\bar{a}ba\bar{b}a$. The length-2 subword counts are $(aa)_w = 2$, $(bb)_w = 1$, $(ab)_w = 1 = (ba)_w$, and $(a\bar{b})_w = 2 = (\bar{b}a)_w$.

Theorem

w is minimal if and only if $|(ab)_w - (a\bar{b})_w| \leq \min((aa)_w, (bb)_w)$.

Representatives of an equivalence class

It is easy to recognize equivalence under

- a permutation ($abab \sim \overline{a}b\overline{a}b \sim baba \sim \cdots$).
- an inner automorphism ($abab \sim baba$).

Equivalence classes

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3.1	aaa
4.1	$aaaa$
4.2	$ab\bar{a}\bar{b}$
4.3	$aabb$ $aba\bar{b}$
5.1	$aaaaa$
5.2	$aaba\bar{b}$
5.3	$aab\bar{a}\bar{b}$
5.4	$aaabb$ $aab\bar{a}b$

6.1	$aaaaaa$
6.2	$aaaba\bar{b}$
6.3	$aaabbb$
6.4	$aaab\bar{a}\bar{b}$
6.5	$aabaa\bar{b}$
6.6	$aaba\bar{b}\bar{b}$
6.7	$aabba\bar{b}$
6.8	$aabb\bar{a}\bar{b}$
6.9	$aab\bar{a}\bar{a}\bar{b}$
6.10	$aaaabb$ $aaab\bar{a}b$ $aab\bar{a}\bar{a}b$
7.1	$aaaaaaa$
7.2	$aaaaba\bar{b}$
7.3	$aaaabbb$

7.4	$aaaab\bar{a}\bar{b}$
7.5	$aaabaa\bar{b}$
7.6	$aaaba\bar{b}\bar{b}$
7.7	$aaabba\bar{b}$
7.8	$aaabb\bar{a}b$
7.9	$aaabb\bar{a}\bar{b}$
7.10	$aaab\bar{a}bb$
7.11	$aaab\bar{a}\bar{a}\bar{b}$
7.12	$aaab\bar{a}\bar{b}\bar{b}$
7.13	$aabaa\bar{b}\bar{b}$
7.14	$aabba\bar{b}\bar{b}$
7.15	$aabb\bar{a}\bar{a}\bar{b}$
7.16	$aaaaabb$ $aaaab\bar{a}b$ $aaab\bar{a}\bar{a}b$

Equivalence classes

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1.1	a
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5.1	$aaaaa$
5.2	$aaba\bar{b}$
5.3	$aab\bar{a}\bar{b}$
5.4	$aaabb$ $aab\bar{a}b$

6.1	$aaaaaa$
6.2	$aaaba\bar{b}$
6.3	$aaabbb$
6.4	$aaab\bar{a}\bar{b}$
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7.2	$aaaaba\bar{b}$
7.3	$aaaabbb$

7.4	$aaaab\bar{a}\bar{b}$
7.5	$aaabaa\bar{b}$
7.6	$aaaba\overline{bb}$
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7.11	$aaab\overline{aa}\bar{b}$
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Growing words

Definition

A child of $w \neq \epsilon$ is a word obtained by duplicating a letter in w. Also define the words $a, \overline{a}, b, \overline{b}$ to be children of ϵ.

Example

The children of *aabb* are *aaabb* and *aabbb*.

A child of a minimal word w is necessarily minimal, since

$$|(ab)_w - (a\overline{b})_w| \leq \min((aa)_w, (bb)_w).$$

Root words

Definition

A root word is a minimal word that is not a child of any minimal word.

Root words are new minimal words with respect to duplicating a letter.

Example

The minimal word $aabb$ is a root word, since neither of its parents abb and aab is minimal.

Characterization of root words

Root words refine the notion of minimal words.

Theorem

w is a root word if and only if $|(ab)_w - (a\bar{b})_w| = (aa)_w = (bb)_w$.

Proof.

Recall: w is minimal if and only if $|(ab)_w - (a\bar{b})_w| \leq \min((aa)_w, (bb)_w)$.
A minimal word w is a root word if and only if replacing any xx by x in w produces a non-minimal word. $\square$

Properties of root words

Theorem

If w is a root word, then $|w|$ is divisible by 4.

The property of being a root word is preserved under automorphisms.

Theorem

If w is a root word, $w \sim v$, and $|w| = |v|$, then v is a root word.

A class W whose minimal words are root words is a root class.

Root classes

0.1	ϵ
1.1	a
2.1	aa
3.1	aaa
4.1	$aaaa$
4.2	$ab\bar{a}\bar{b}$
4.3	$aabb$ $aba\bar{b}$
5.1	$aaaaa$
5.2	$aaba\bar{b}$
5.3	$aab\bar{a}\bar{b}$
5.4	$aaabb$ $aab\bar{a}b$
6.1	$aaaaaa$
6.2	$aaaba\bar{b}$
6.3	$aaabbb$
6.4	$aaab\bar{a}\bar{b}$
6.5	$aabaa\bar{b}$
6.6	$aaba\overline{bb}$
6.7	$aabba\bar{b}$
6.8	$aabb\bar{a}\bar{b}$
6.9	$aab\overline{aa}\bar{b}$
6.10	$aaaabb$ $aaab\bar{a}b$ $aab\overline{aa}b$
7.1	$aaaaaaa$
7.2	$aaaaba\bar{b}$
7.3	$aaaabbb$
7.4	$aaaab\bar{a}\bar{b}$
7.5	$aaabaa\bar{b}$
7.6	$aaaba\overline{bb}$
7.7	$aaabba\bar{b}$
7.8	$aaabb\bar{a}b$
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7.14	$aabba\overline{bb}$
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7.16	$aaaaabb$ $aaaab\bar{a}b$ $aaab\overline{aa}b$

Outline

Equivalence class graph

Khan (2004) gave a structural description of equivalence classes of F_2.

A shorter classification distinguishes non-root from root classes.

Let $\Gamma(W)$ be a directed graph whose vertices are the minimal words of W up to permutations and inner automorphisms.

We draw an edge $w \to v$ for each one-letter automorphism ϕ such that $\phi(w)$ and v differ only by a permutation and an inner automorphism.

Example

Consider the equivalence class W containing $aabb$ and $aba\bar{b}$.
Applying the 4 one-letter automorphisms to $aabb$...

$$
\begin{array}{llcll}
a \mapsto ab: & ababbb & \qquad & b \mapsto ba: & aababa \\
a \mapsto a\bar{b}: & a\bar{b}ab & \qquad & b \mapsto b\bar{a}: & ab\bar{a}b
\end{array}
$$

Applying the 4 one-letter automorphisms to $aba\bar{b}$...

$$
\begin{array}{llcll}
a \mapsto ab: & abba & \qquad & b \mapsto ba: & aba\bar{b} \\
a \mapsto a\bar{b}: & aa\overline{bb} & \qquad & b \mapsto b\bar{a}: & aba\bar{b}
\end{array}
$$

Therefore $\Gamma(W)$ is

$$
\begin{tikzcd}
aabb \arrow[r, bend left=10] \arrow[r, leftrightarrow] & aba\bar{b} \arrow[l, bend left=10] \arrow[loop above] \arrow[loop below]
\end{tikzcd}.
$$

By Whitehead's theorem, $\Gamma(W)$ is connected.

Structure of non-root classes

Theorem

Let W be a non-root class. Then $\Gamma(W)$ has one of the following forms.

(i)

(ii)

(iii)

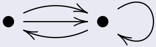
Structure of root classes

Theorem

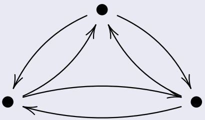
Let W be a root class with no alternating minimal word. Then $\Gamma(W)$ is one of the following graphs.

(i)

(ii)



(iii)

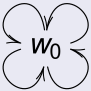


Structure of root classes

Theorem

Let W be a root class containing an alternating minimal word w_0. Then $\Gamma(W)$ is one of the following graphs.

(i)



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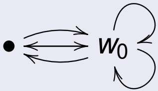

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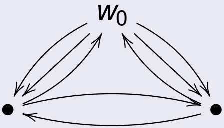

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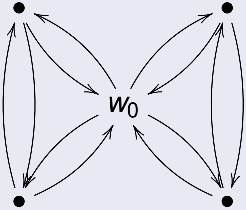

Conclusion

Moreover, each of these forms occurs.

- Non-root classes have simple structure.
- If W is a root class, $\Gamma(W)$ is one of seven graphs.

Corollary

If W is a root class, then $\Gamma(W)$ has 1, 2, 3, or 5 vertices.

Future work:
Understand descendancy on the level of equivalence classes.
Can equivalence classes be enumerated?