Congruences for diagonals of rational power series

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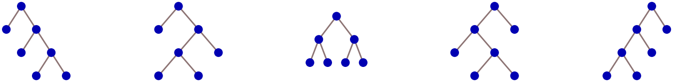
Catalan numbers modulo 2

What do combinatorial sequences look like modulo p^α?

$C(n)_{n\geq 0} = 1, 1, 2, 5, 14, 42, 132, 429, \ldots$



$C(3) = 5$

$(C(n) \bmod 2)_{n\geq 0} = 1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, \ldots$

Theorem (folklore)

For all $n \geq 0$, $C(n)$ is odd if and only if $n + 1$ is a power of 2.

Catalan numbers modulo 4 and 8

Theorem (Eu–Liu–Yeh 2008)

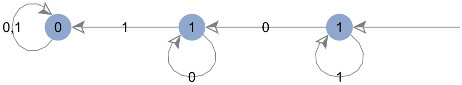
For all $n \geq 0$,

$$C(n) \bmod 4 = \begin{cases} 1 & \text{if } n+1 = 2^a \text{ for some } a \geq 0 \\ 2 & \text{if } n+1 = 2^b + 2^a \text{ for some } b > a \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.2. *Let C_n be the nth Catalan number. First of all, $C_n \not\equiv_8 3$ and $C_n \not\equiv_8 7$ for any n. As for other congruences, we have*

$$C_n \equiv_8 \begin{cases} 1 & \text{if } n = 0 \text{ or } 1; \\ 2 & \text{if } n = 2^a + 2^{a+1} - 1 \text{ for some } a \geq 0; \\ 4 & \text{if } n = 2^a + 2^b + 2^c - 1 \text{ for some } c > b > a \geq 0; \\ 5 & \text{if } n = 2^a - 1 \text{ for some } a \geq 2; \\ 6 & \text{if } n = 2^a + 2^b - 1 \text{ for some } b - 2 \geq a \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

$C(n)$ is odd if and only if $n+1$ is a power of 2.



This automaton outputs $C(n) \bmod 2$ when fed the base-2 digits of n, starting with the least significant digit.

$(C(n) \bmod 2)_{n\geq 0}$ is 2-automatic.

Let $\mathcal{D}f$ denote the diagonal of a multivariate formal power series f.

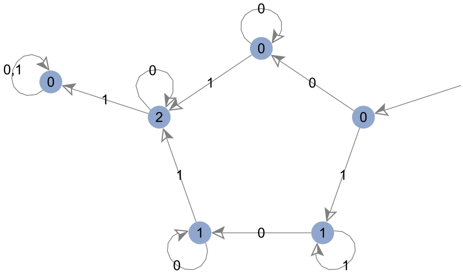
Theorem (Denef–Lipshitz 1987)

Let $\alpha \geq 1$. Let $P(\mathbf{x}), Q(\mathbf{x}) \in \mathbb{Z}_p[\mathbf{x}]$ such that $Q(0,\dots,0) \not\equiv 0 \mod p$. Then the coefficient sequence of $\left(\mathcal{D}\,\frac{P(\mathbf{x})}{Q(\mathbf{x})}\right) \bmod p^\alpha$ is p-automatic.

Catalan numbers modulo 4

$\sum_{n \geq 1} C(n)x^n$ is the diagonal of

$$\frac{y(2xy^2 + 2xy - 1)}{xy^2 + 2xy + x - 1}.$$



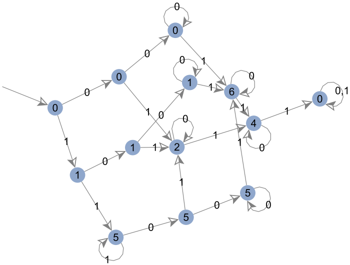
By computing an automaton for a sequence $\bmod p^\alpha$, we can answer...

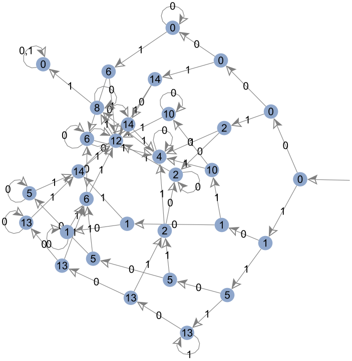
- Are there forbidden residues?
- What is the limiting distribution of residues (if it exists)?
- Is the sequence eventually periodic?
- Many other questions known to be decidable.

Catalan numbers modulo 8 and 16

Theorem (Liu–Yeh)

For all $n \geq 0$, $C(n) \not\equiv 9 \mod 16$.





Catalan numbers modulo 2^α

Theorem

For all $n \geq 0$,

- $C(n) \not\equiv 17, 21, 26 \mod 32$,
- $C(n) \not\equiv 10, 13, 33, 37 \mod 64$,
- $C(n) \not\equiv 18, 54, 61, 65, 66, 69, 98, 106, 109 \mod 128$.

Only $\approx$ 35% of the residues modulo 512 are attained by some $C(n)$.

Open question

Does the fraction of residues modulo 2^α that are attained by some Catalan number tend to 0 as α gets large?

Apéry numbers

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$ arose in Apéry's proof that $\zeta(3)$ is irrational.

$A(n)_{n\geq 0} = 1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$

Straub (2014): $\sum_{n\geq 0} A(n) x^n$ is the diagonal of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}.$$

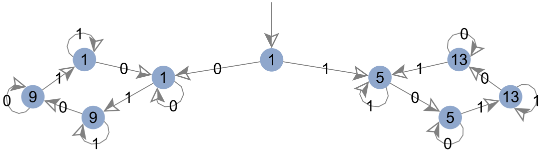
Computing automata allowed us to resolve some conjectures.

Apéry numbers modulo 16

Gessel (1982) proved a conjecture of Chowla–Cowles–Cowles that

$$A(n) \bmod 8 = \begin{cases} 1 & \text{if } n \text{ is even} \\ 5 & \text{if } n \text{ is odd.} \end{cases}$$

Gessel asked whether $A(n)$ is periodic modulo 16.



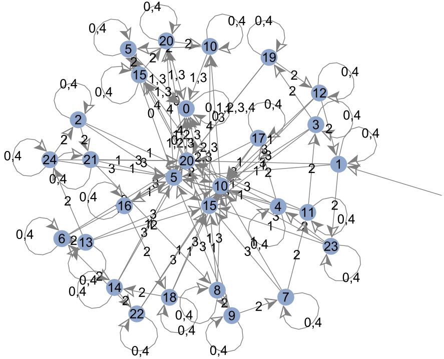
Theorem

$(A(n) \bmod 16)_{n \geq 0}$ *is not eventually periodic.*

Apéry numbers modulo 25

Theorem (special case of a conjecture of Beukers (1995))

If there are at least two 1*s and* 3*s in the base-*5 *representation of* n, *then* $A(n) \equiv 0 \mod 5^2$.



Apéry numbers modulo 25

Theorem

Let $e_2(n)$ be the number of 2s in the base-5 representation of n. If n contains no 1 or 3 in base 5, then $A(n) \equiv (-2)^{e_2(n)} \mod 25$.