Congruences for diagonals of rational power series

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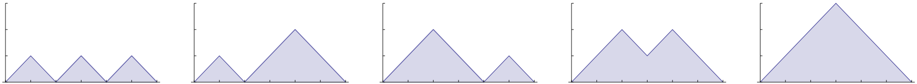
Outline

1. Algebraic sequences
2. Automatic sequences
3. Diagonals of rational power series
4. Congruence gallery

Algebraic sequences

A sequence $(a_n)_{n\geq 0}$ of integers is algebraic if its generating function $\sum_{n\geq 0} a_n x^n$ is algebraic over $\mathbb{Q}(x)$.

Catalan numbers $C(n)_{n\geq 0} = 1, 1, 2, 5, 14, 42, 132, 429, \ldots$ [A000108]

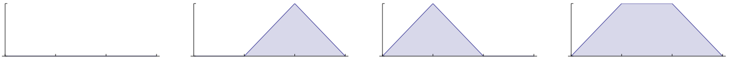


$C(3) = 5$
$C(n) = \frac{1}{n+1}\binom{2n}{n}$

$$y = \sum_{n\geq 0} C(n)x^n = \frac{1-\sqrt{1-4x}}{2x} \text{ satisfies } xy^2 - y + 1 = 0.$$

Motzkin numbers

Motzkin numbers $M(n)_{n\geq 0} = 1, 1, 2, 4, 9, 21, 51, 127, \ldots$ [A001006]



$M(3) = 4$

$y = \sum_{n\geq 0} M(n)x^n$ satisfies $x^2y^2 + (x-1)y + 1 = 0$.

Other algebraic sequences:

- sequence of Fibonacci numbers, etc.
- number of binary trees avoiding a pattern
- number of planar maps with n vertices

Arithmetic properties

Let p^α be a prime power.

Question

If $(a_n)_{n\geq 0}$ is algebraic, what does $(a_n \bmod p^\alpha)_{n\geq 0}$ look like?

Deutsch and Sagan (2006) studied Catalan and Motzkin numbers, Riordan numbers, central binomial and trinomial coefficients, etc.

$$C(n)_{n\geq 0} = 1, 1, 2, 5, 14, 42, 132, 429, \dots$$
$$(C(n) \bmod 2)_{n\geq 0} = 1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, \dots$$

Theorem

For all $n \geq 0$, $C(n)$ is odd if and only if $n+1$ is a power of 2.

Deutsch and Sagan gave a combinatorial proof.

Motzkin numbers modulo 8

$M(n)_{n\geq 0} = 1, 1, 2, 4, 9, 21, 51, 127, \ldots$ [A001006]

Deutsch, Sagan, and Amdeberhan conjectured necessary and sufficient conditions for $M(n)$ to be divisible by 4.

…and that no Motzkin number is divisible by 8.

Theorem (Eu–Liu–Yeh 2008)

For all $n \geq 0$, $M(n) \not\equiv 0 \mod 8$.

Catalan numbers modulo 4

To prove this, Eu, Liu, and Yeh determined $C(n) \bmod 4 \ldots$

Theorem (Eu–Liu–Yeh)

For all $n \geq 0$,

$$C(n) \bmod 4 = \begin{cases} 1 & \text{if } n = 2^a - 1 \text{ for some } a \geq 0 \\ 2 & \text{if } n = 2^b + 2^a - 1 \text{ for some } b > a \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $C(n) \not\equiv 3 \mod 4$ for all $n \geq 0$.

Catalan numbers modulo 8

…and $C(n)$ mod 8:

Theorem 4.2. *Let C_n be the nth Catalan number. First of all, $C_n \not\equiv_8 3$ and $C_n \not\equiv_8 7$ for any n. As for other congruences, we have*

$$C_n \equiv_8 \begin{cases} 1 & \text{if } n = 0 \text{ or } 1; \\ 2 & \text{if } n = 2^a + 2^{a+1} - 1 \text{ for some } a \geq 0; \\ 4 & \text{if } n = 2^a + 2^b + 2^c - 1 \text{ for some } c > b > a \geq 0; \\ 5 & \text{if } n = 2^a - 1 \text{ for some } a \geq 2; \\ 6 & \text{if } n = 2^a + 2^b - 1 \text{ for some } b - 2 \geq a \geq 0; \\ 0 & \text{otherwise}. \end{cases}$$

Catalan numbers modulo 16

Liu and Yeh (2010) determined $C(n)$ mod 16:

Theorem 5.5. *Let c_n be the n-th Catalan number. First of all, $c_n \not\equiv_{16} 3, 7, 9, 11, 15$ for any n. As for the other congruences, we have*

$$c_n \equiv_{16} \begin{cases} \left.\begin{array}{c} 1 \\ 5 \\ 13 \end{array}\right\} & \textit{if} \quad d(\alpha) = 0 \ \textit{and} \quad \begin{cases} \beta \le 1, \\ \beta = 2, \\ \beta \ge 3, \end{cases} \\ \left.\begin{array}{c} 2 \\ 10 \end{array}\right\} & \textit{if} \quad d(\alpha) = 1,\ \alpha = 1 \ \textit{and} \quad \begin{cases} \beta = 0 \ \textit{or} \ \beta \ge 2, \\ \beta = 1, \end{cases} \\ \left.\begin{array}{c} 6 \\ 14 \end{array}\right\} & \textit{if} \quad d(\alpha) = 1,\ \alpha \ge 2 \ \textit{and} \quad \begin{cases} (\alpha = 2, \beta \ge 2) \ \textit{or} \ (\alpha \ge 3, \beta \le 1), \\ (\alpha = 2, \beta \le 1) \ \textit{or} \ (\alpha \ge 3, \beta \ge 2), \end{cases} \\ \left.\begin{array}{c} 4 \\ 12 \end{array}\right\} & \textit{if} \quad d(\alpha) = 2 \ \textit{and} \quad \begin{cases} zr(\alpha) \equiv_2 0, \\ zr(\alpha) = 1, \end{cases} \\ \quad 8 & \textit{if} \quad d(\alpha) = 3, \\ \quad 0 & \textit{if} \quad d(\alpha) \ge 4. \end{cases}$$

where $\alpha = (CF_2(n+1) - 1)/2$ and $\beta = \omega_2(n+1)$ (or $\beta = \min\{i \mid n_i = 0\}$).

They also determined $C(n)$ mod 64.

Questions

$C(n)$ mod 2^α seems to reflect the base-2 digits of n.

Does this hold for other combinatorial sequences modulo p^α?

Are piecewise functions the best notation?

Power series congruences

Kauers, Krattenthaler, and Müller developed a systematic method for producing congruences modulo 2^α (2012) and modulo 3^α (2013).

Let $\Phi(z) = \sum_{n \geq 0} z^{2^n}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{Cat}_n\, z^n &= 32z^5 + 16z^4 + 6z^2 + 13z + 1 + \left(32z^4 + 32z^3 + 20z^2 + 44z + 40\right)\Phi(z)\\ &+ \left(16z^3 + 56z^2 + 30z + 52 + \frac{12}{z}\right)\Phi^2(z) + \left(32z^3 + 60z + 60 + \frac{28}{z}\right)\Phi^3(z)\\ &+ \left(32z^3 + 16z^2 + 48z + 18 + \frac{35}{z}\right)\Phi^4(z) + \left(32z^2 + 44\right)\Phi^5(z)\\ &+ \left(48z + 8 + \frac{50}{z}\right)\Phi^6(z) + \left(32z + 32 + \frac{4}{z}\right)\Phi^7(z) \qquad \text{modulo } 64\end{aligned}$$

Outline

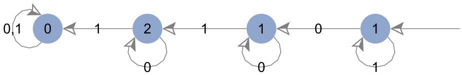
Catalan numbers modulo 4

Theorem (Eu–Liu–Yeh)

For all $n \geq 0$,

$$C(n) \bmod 4 = \begin{cases} 1 & \text{if } n = 2^a - 1 \text{ for some } a \geq 0 \\ 2 & \text{if } n = 2^b + 2^a - 1 \text{ for some } b > a \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Process the binary digits of n, starting with the least significant digit.



This machine is a deterministic finite automaton with output (DFAO).

Automatic sequences

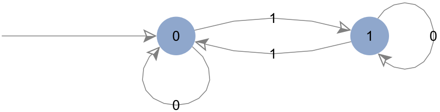
A sequence $(a_n)_{n\geq 0}$ is k-automatic if there is DFAO whose output is a_n when fed the base-k digits of n.

$(C(n) \bmod 4)_{n\geq 0} = 1, 1, 2, 1, 2, 2, 0, 1, \ldots$ is 2-automatic.

Let $T(n) = $ (number of 1s in the binary representation of n) mod 2.
The Thue–Morse sequence

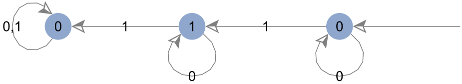
$$T(n)_{n\geq 0} = 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \ldots$$

is 2-automatic. It is also cube-free.

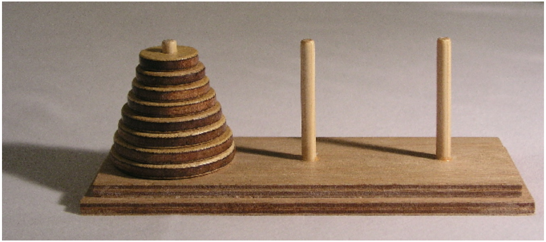


Examples of 2-automatic sequences

- Characteristic sequence of powers of 2:



- Minimal solution to the “infinite” tower of Hanoi puzzle

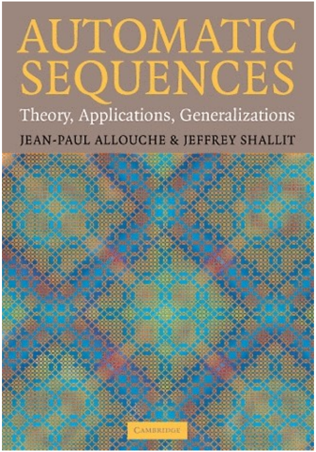


Automatic sequences

Automatic sequences have been studied extensively.

Büchi 1960:
Every eventually periodic sequence is k-automatic for every $k \geq 2$.

Several natural characterizations of automatic sequences are known.



Algebraic characterization

Theorem (Christol–Kamae–Mendès France–Rauzy 1980)

Let $(a_n)_{n\geq 0}$ be a sequence of elements in $\mathbb{F}_p$. Then $(a_n)_{n\geq 0}$ is p-automatic if and only if $\sum_{n\geq 0} a_n x^n$ is algebraic over $\mathbb{F}_p(x)$.

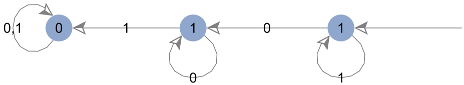
Algebraic sequences of integers modulo p are p-automatic.

$y = 1 + 1x + 0x^2 + 1x^3 + 0x^4 + 0x^5 + 0x^6 + \cdots$ satisfies

$$xy^2 + y + 1 = 0$$

in $\mathbb{F}_2[\![x]\!]$.

The proof is constructive.



Prime powers?

Outline

Converting algebraic to rational

The diagonal of a formal power series is

$$\mathcal{D}\left(\sum_{n,m\geq 0} a_{n,m}x^n y^m\right) := \sum_{n\geq 0} a_{n,n}x^n.$$

Algebraic sequences can be realized as diagonals of rational functions.

Proposition (Furstenberg 1967)

Let $P(x,y) \in \mathbb{Q}[x,y]$ such that $\frac{\partial P}{\partial y}(0,0) \neq 0$.
If $f(x) \in \mathbb{Q}[\![x]\!]$ is a power series with $f(0) = 0$ and $P(x, f(x)) = 0$, then

$$f(x) = \mathcal{D}\left(\frac{y\frac{\partial P}{\partial y}(xy,y)}{P(xy,y)/y}\right).$$

Catalan numbers

$$y = \sum_{n \geq 0} C(n) x^n = \frac{1 - \sqrt{1 - 4x}}{2x} \text{ satisfies } xy^2 - y + 1 = 0.$$

Since $C(0) = 1 \neq 0$, consider $y = 0 + \sum_{n \geq 1} C(n) x^n$, which satisfies

$$P(x, y) := x(y+1)^2 - (y+1) + 1 = 0.$$

Then $\frac{\partial P}{\partial y}(0,0) = -1 \not\equiv 0 \mod 2$, so $\sum_{n \geq 1} C(n) x^n$ is the diagonal of

$$\begin{aligned}
\frac{y(2xy^2 + 2xy - 1)}{xy^2 + 2xy + x - 1} = \quad & 0x^0y^0 + 1x^0y + 0x^0y^2 + 0x^0y^3 + 0x^0y^4 + 0x^0y^5 + \cdots \\
& + 0x^1y^0 + 1x^1y + 0x^1y^2 - 1x^1y^3 + 0x^1y^4 + 0x^1y^5 + \cdots \\
& + 0x^2y^0 + 1x^2y + 2x^2y^2 + 0x^2y^3 - 2x^2y^4 - 1x^2y^5 + \cdots \\
& + 0x^3y^0 + 1x^3y + 4x^3y^2 + 5x^3y^3 + 0x^3y^4 - 5x^3y^5 + \cdots \\
& + 0x^4y^0 + 1x^4y + 6x^4y^2 + 14x^4y^3 + 14x^4y^4 + 0x^4y^5 + \cdots \\
& + 0x^5y^0 + 1x^5y + 8x^5y^2 + 27x^5y^3 + 48x^5y^4 + 42x^5y^5 + \cdots \\
& + \cdots .
\end{aligned}$$

Automata for diagonals of rational power series

Theorem (Denef–Lipshitz 1987)

Let $R(x, y)$ and $Q(x, y)$ be polynomials in $\mathbb{Z}_p[x, y]$ such that $Q(0, 0) \not\equiv 0 \mod p$, and let $\alpha \geq 1$.
Then the coefficient sequence of

$$\mathcal{D}\left(\frac{R(x, y)}{Q(x, y)}\right) \bmod p^\alpha$$

is p-automatic.

Here $\mathbb{Z}_p$ denotes the set of p-adic integers.

Algorithm

Let $0 \leq d \leq p-1$.
The Cartier operator is the map on $\mathbb{Z}_p[\![x,y]\!]$ defined by

$$\Lambda_{d,d}\left(\sum_{n,m\geq 0} a_{n,m} x^n y^m\right) := \sum_{n,m\geq 0} a_{pn+d,pm+d} x^n y^m.$$

To compute an automaton for the coefficients of $\mathcal{D}\left(\frac{R(x,y)}{Q(x,y)}\right)$ mod p^α:

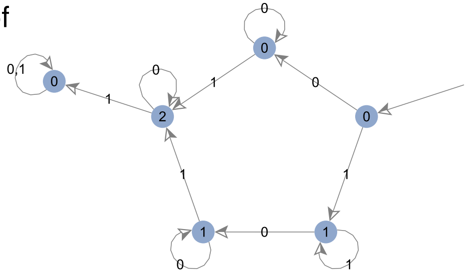
1. Compute the image of $\frac{R(x,y)}{Q(x,y)} = \frac{R(x,y)\cdot Q(x,y)^{p^{\alpha-1}-1}}{Q(x,y)^{p^{\alpha-1}}}$ under each $\Lambda_{d,d}$.
2. Draw an edge labeled d from $\frac{s(x,y)}{Q(x,y)^{p^{\alpha-1}}}$ to $\frac{t(x,y)}{Q(x,y)^{p^{\alpha-1}}}$ if

$$\Lambda_{d,d}\left(\frac{s(x,y)}{Q(x,y)^{p^{\alpha-1}}}\right) = \frac{t(x,y)}{Q(x,y)^{p^{\alpha-1}}}.$$

3. Iterate, and stop when all images have been computed.

Catalan numbers modulo 4

$\sum_{n \geq 1} C(n)x^n$ is the diagonal of
$$\frac{y(2xy^2 + 2xy - 1)}{xy^2 + 2xy + x - 1}.$$



By computing an automaton for a sequence $\bmod p^\alpha$, we can answer...

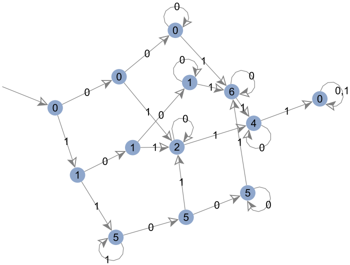
- Are there forbidden residues?
- What is the limiting distribution of residues (if it exists)?
- Is the sequence eventually periodic?

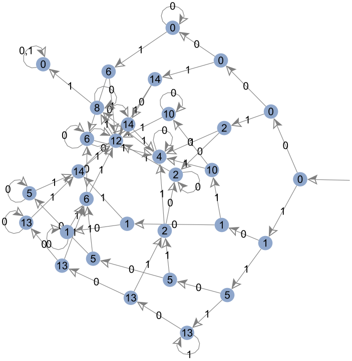
Outline

Catalan numbers modulo 8 and 16

Theorem (Liu–Yeh)

For all $n \geq 0$, $C(n) \not\equiv 9 \mod 16$.





Catalan numbers modulo 2^α

Theorem

For all $n \geq 0$*,*

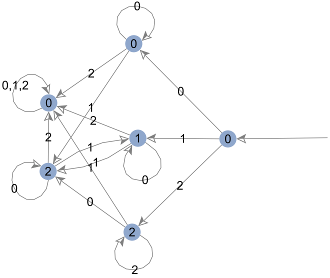
- $C(n) \not\equiv 17, 21, 26 \mod 32$*,*
- $C(n) \not\equiv 10, 13, 33, 37 \mod 64$*,*
- $C(n) \not\equiv 18, 54, 61, 65, 66, 69, 98, 106, 109 \mod 128$*,*
- $C(n) \not\equiv 22, 34, 45, 62, 82, 86, 118, 129, 130, 133, 157, 170, 178, 253 \mod 256$*.*

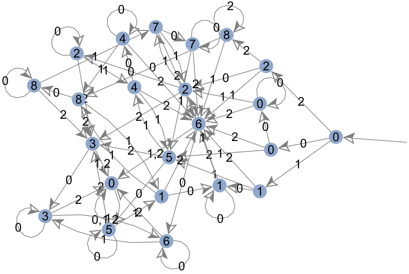
Only $\approx 35\%$ of the residues modulo 512 are attained by some $C(n)$.

Open question

Does the fraction of residues modulo 2^α *that are attained by some Catalan number tend to* 0 *as* α *gets large?*

Catalan numbers modulo 3^α





There are no known forbidden residues modulo 3^α.

Open question

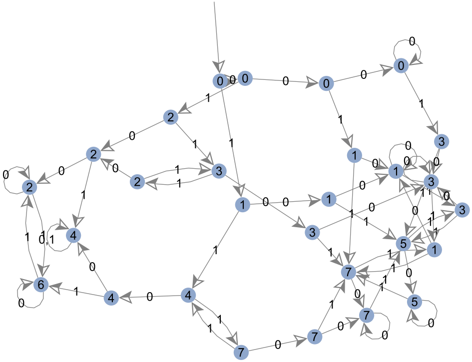
Do there exist α and r such that $C(n) \not\equiv r \mod 3^\alpha$ for all $n \geq 0$?

Motzkin numbers modulo 8

Theorem (Eu–Liu–Yeh)

For all $n \geq 0$, $M(n) \not\equiv 0 \mod 8$.

Proof:



Motzkin numbers modulo p^2

Theorem

For all $n \geq 0$, $M(n) \not\equiv 0 \mod 5^2$.

(2 seconds; 144 states)

Theorem

For all $n \geq 0$, $M(n) \not\equiv 0 \mod 13^2$.

(10 minutes; 2125 states)

Conjecture

Let $p \in \{31, 37, 61\}$. For all $n \geq 0$, $M(n) \not\equiv 0 \mod p^2$.

Open question

Are there infinitely many p such that $M(n) \not\equiv 0 \mod p^2$ for all $n \geq 0$?

A few more well-known sequences

Riordan numbers: $R(n)_{n \geq 0} = 1, 0, 1, 1, 3, 6, 15, 36, \ldots$ [A005043]

Theorem

For all $n \geq 0$, $R(n) \not\equiv 16 \mod 32$.

Number of directed animals:
$P(n)_{n \geq 0} = 1, 1, 2, 5, 13, 35, 96, 267, \ldots$ [A005773]

Theorem

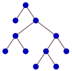
For all $n \geq 0$, $P(n) \not\equiv 16 \mod 32$.

Number of restricted hexagonal polyominoes:
$H(n)_{n \geq 0} = 1, 1, 3, 10, 36, 137, 543, 2219, \ldots$ [A002212]

Theorem

For all $n \geq 0$, $H(n) \not\equiv 0 \mod 8$.

Binary trees avoiding a pattern

Let a_n be the number of $(n+1)$-leaf binary trees avoiding  .

$(a_n)_{n\geq 0} = 1, 1, 2, 5, 14, 41, 124, 385, \ldots$ [A159771]

The generating function satisfies

$$2x^2y^2 - (3x^2 - 2x + 1)y + x^2 - x + 1 = 0.$$

Theorem

For all $n \geq 0$,

$$\begin{aligned} a_n &\not\equiv 3 \mod 4, \\ a_n &\not\equiv 13 \mod 16, \\ a_n &\not\equiv 21 \mod 32, \\ a_n &\not\equiv 37 \mod 64. \end{aligned}$$

Permutations avoiding a pair of patterns

Let a_n be the number of permutations of length n avoiding 3412 and 2143.

$(a_n)_{n\geq 0} = 1, 1, 2, 6, 22, 86, 340, 1340, \ldots$ [A029759]

Atkinson (1998) showed that $\sum_{n\geq 0} a_n x^n$ is algebraic.

Theorem

For all $n \geq 0$,

$$\begin{aligned}
a_n &\not\equiv 10, 14 && \mod 16,\\
a_n &\not\equiv 18 && \mod 32,\\
a_n &\not\equiv 34, 54 && \mod 64,\\
a_n &\not\equiv 44, 66, 102 && \mod 128,\\
a_n &\not\equiv 20, 130, 150, 166, 188, 204, 212, 214, 220, 236, 252 && \mod 256.
\end{aligned}$$

Apéry numbers

$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ arose in Apéry's proof that $\zeta(3)$ is irrational.

$A(n)_{n \geq 0} = 1, 5, 73, 1445, 33001, 819005, 21460825, \ldots$ [A005259]

Straub (2014): $\sum_{n \geq 0} A(n) x^n$ is the diagonal of

$$\frac{1}{(1 - x_1 - x_2)(1 - x_3 - x_4) - x_1 x_2 x_3 x_4}.$$

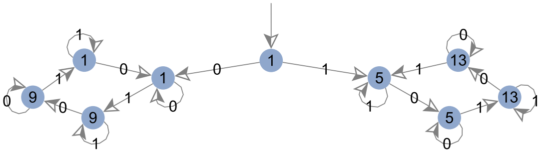
Computing automata allows us to resolve some conjectures.

Apéry numbers modulo 16

Chowla, Cowles, and Cowles conjectured, and Gessel (1982) proved,

$$A(n) \bmod 8 = \begin{cases} 1 & \text{if } n \text{ is even} \\ 5 & \text{if } n \text{ is odd.} \end{cases}$$

Gessel asked whether $A(n)$ is periodic modulo 16.



Theorem

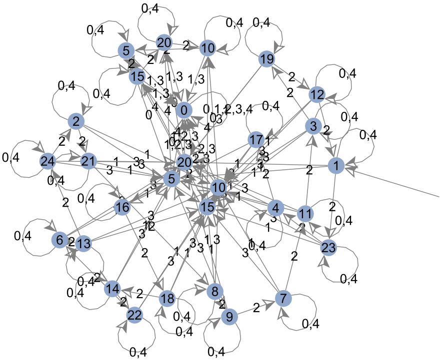
The sequence $(A(n) \bmod 16)_{n \geq 0}$ *is not eventually periodic.*

Apéry numbers modulo 25

Beukers (1995) conjectured that if there are α 1s and 3s in the standard base-5 representation of n then $A(n) \equiv 0 \mod 5^\alpha$. (Proved recently by Delaygue.)

Theorem

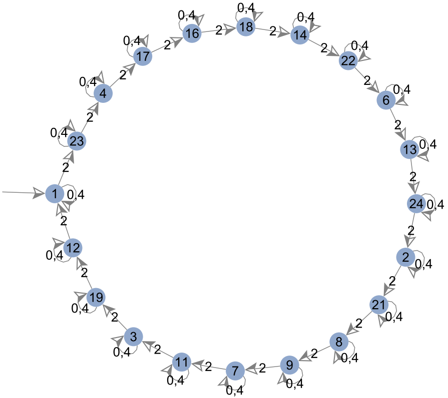
Beukers' conjecture is true for $\alpha = 2$.



Apéry numbers modulo 25

Theorem

Let $e_2(n)$ be the number of 2s in the standard base-5 representation of n. If n contains no 1 or 3 in base 5, then $A(n) \equiv (-2)^{e_2(n)} \mod 25$.



Scope

Christol (1990) conjectured that if $(a_n)_{n \geq 0}$ is an integer sequence which

- is holonomic (satisfies a linear recurrence with polynomial coefficients) and
- grows at most exponentially,

then $(a_n)_{n \geq 0}$ is the diagonal of a rational function.

$(n!)_{n \geq 0}$ grows too quickly to be the diagonal of a rational function.

If the conjecture is true, then essentially every sequence that occurs in combinatorics is p-automatic when reduced modulo p^α.

Symbolic p^α

Write $n = n_\ell \cdots n_1 n_0$ and $m = m_\ell \cdots m_1 m_0$ in base p.
Lucas' theorem:

$$\binom{n}{m} \equiv \prod_{i=0}^{\ell} \binom{n_i}{m_i} \mod p.$$

For the Apéry numbers, Gessel (1982) proved

$$A(n) \equiv \prod_{i=0}^{\ell} A(n_i) \mod p.$$

Our method doesn't allow α to vary (for fixed p).