Combinatorial structure behind Sinkhorn limits

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Setup

Given a square matrix with positive entries, turn it into a "close" doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows . . .

$$\begin{bmatrix} .800000 & .200000 \\ .666667 & .333333 \end{bmatrix}$$

Setup

Given a square matrix with positive entries, turn it into a "close" doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns . . .

$$\begin{bmatrix} .545455 & .375000 \\ .454545 & .625000 \end{bmatrix}$$

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$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows ...

$$\begin{bmatrix} .592593 & .407407 \\ .421053 & .578947 \end{bmatrix}$$

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Given a square matrix with positive entries, turn it into a "close" doubly stochastic matrix of the same size (row and column sums are 1).

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .584615 & .413043 \\ .415385 & .586957 \end{bmatrix}$$

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$$\begin{bmatrix} .585987 & .414013 \\ .414414 & .585586 \end{bmatrix}$$

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Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585752 & .414179 \\ .414248 & .585821 \end{bmatrix}$$

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Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585792 & .414208 \\ .414219 & .585781 \end{bmatrix}$$

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Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585785 & .414213 \\ .414215 & .585787 \end{bmatrix}$$

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In the limit, we obtain the Sinkhorn limit.

The limit exists: Sinkhorn 1964.

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Question

What are the entries?

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Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585786 & .414214 \\ .414214 & .585786 \end{bmatrix} \approx \begin{bmatrix} 2-\sqrt{2} & -1+\sqrt{2} \\ -1+\sqrt{2} & 2-\sqrt{2} \end{bmatrix}$$

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What are the entries?

Notation:

$$\textcolor{red}{\mathrm{Sink}}\left(\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2-\sqrt{2} & -1+\sqrt{2} \\ -1+\sqrt{2} & 2-\sqrt{2} \end{bmatrix}$$

Theorem (Nathanson 2020)

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with positive entries,

$$\mathrm{Sink}(A) = \frac{1}{\sqrt{ad}+\sqrt{bc}} \begin{bmatrix} \sqrt{ad} & \sqrt{bc} \\ \sqrt{bc} & \sqrt{ad} \end{bmatrix}.$$

For a symmetric 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ with positive entries:

Theorem (Ekhad–Zeilberger 2019)

The top-left entry x of $\mathrm{Sink}(A)$ *satisfies* $c_4 x^4 + \cdots + c_1 x + c_0 = 0$, *where*

$$
\begin{aligned}
c_4 &= -(a_{12}^2 - a_{11}a_{22})(a_{13}^2 - a_{11}a_{33})(-a_{11}a_{22}a_{33} + a_{11}a_{23}^2 + a_{12}^2 a_{33} - 2a_{12}a_{13}a_{23} + a_{13}^2 a_{22}) \\
c_3 &= (-4a_{11}^3 a_{22}^2 a_{33}^2 + 4a_{11}^3 a_{22} a_{23}^2 a_{33} + 4a_{11}^2 a_{12}^2 a_{22} a_{33}^2 - 3a_{11}^2 a_{12}^2 a_{23}^2 a_{33} - 2a_{11}^2 a_{12} a_{13} a_{22} a_{23} a_{33} + 4a_{11}^2 a_{13}^2 a_{22}^2 a_{33} \\
&\quad - 3a_{11}^2 a_{13}^2 a_{22} a_{23}^2 - 2a_{11} a_{12}^2 a_{13}^2 a_{22} a_{33} + 2a_{11} a_{12}^2 a_{13}^2 a_{23}^2 - a_{12}^4 a_{13}^2 a_{33} + 2a_{12}^3 a_{13}^3 a_{23} - a_{12}^2 a_{13}^4 a_{22}) \\
c_2 &= a_{11}(6a_{11}^2 a_{22}^2 a_{33}^2 - 6a_{11}^2 a_{22} a_{23}^2 a_{33} - 2a_{11} a_{12}^2 a_{22} a_{33}^2 + 3a_{11} a_{12}^2 a_{23}^2 a_{33} - 2a_{11} a_{12} a_{13} a_{22} a_{23} a_{33} - 2a_{11} a_{13}^2 a_{22}^2 a_{33} \\
&\quad + 3a_{11} a_{13}^2 a_{22} a_{23}^2 + 2a_{12}^3 a_{13} a_{23} a_{33} - 3a_{12}^2 a_{13}^2 a_{22} a_{33} - a_{12}^2 a_{13}^2 a_{23}^2 + 2a_{12} a_{13}^3 a_{22} a_{23}) \\
c_1 &= -a_{11}^2 (4a_{11} a_{22}^2 a_{33}^2 - 4a_{11} a_{22} a_{23}^2 a_{33} + a_{12}^2 a_{23}^2 a_{33} - 2a_{12} a_{13} a_{22} a_{23} a_{33} + a_{13}^2 a_{22} a_{23}^2) \\
c_0 &= a_{11}^3 a_{22} a_{33} (a_{22} a_{33} - a_{23}^2)
\end{aligned}
$$

Computed with Gröbner bases.

The entries are algebraic with degree at most 4.

For general 3×3 matrices, the Sinkhorn limit wasn't known!

$$\operatorname{Sink}\left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix}\right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

Can we identify these numbers?

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Compute the top-left entry to high precision:
$x \approx .250338374059368489454547286851429252833867221735301677 1994$

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Compute the top-left entry to high precision:
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Assume it's algebraic. Guess the degree:

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Assume it's algebraic. Guess the degree: 4

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Assume it's algebraic. Guess the degree: 4
Use PSLQ to find a likely polynomial:

$$b_4 x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0 \approx 0$$

For general 3×3 matrices, the Sinkhorn limit wasn't known!

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Compute the top-left entry to high precision:
$x \approx .25033837405936848945454728685142925283386722173530167719940$

Assume it's algebraic. Guess the degree: 4
Use PSLQ to find a likely polynomial:

$$36164989943x^4 + 333428071444x^3 + 65054452280x^2 - 41075578985x + 832844043 \approx 0$$

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garbage!

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Compute the top-left entry to high precision:
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Assume it's algebraic. Guess the degree: 5
Use PSLQ to find a likely polynomial:

$$860608662x^5 - 311838369x^4 + 602643248x^3 + \cdots - 1242712455x + 300075073 \approx 0$$

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Assume it's algebraic. Guess the degree: 6
Use PSLQ to find a likely polynomial:

$$236379x^6 + 502124x^5 - 1610856x^4 + 19808x^3 + 661120x^2 - 94592x - 12288 = 0$$

For general 3×3 matrices, the Sinkhorn limit wasn't known!

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Conjecture (Chen and Varghese 2019, Hofstra SSRP)

For 3×3 matrices A, the entries of $\mathrm{Sink}(A)$ have degree at most 6.

It suffices to describe the top-left entry of $\mathrm{Sink}(A)$.

Fact

If we know one entry of $\mathrm{Sink}(A)$ *as a function of* A*, then we know them all.*

Reason: Iterative scaling isn't sensitive to row or column order.

For a 3×3 matrix, what is the top-left entry of $\mathrm{Sink}(A)$? System of equations...

Row scaling — multiplication on the left.
Column scaling — multiplication on the right.

$$\mathrm{Sink}(A) = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \qquad R = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad C = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

9 equations from $\mathrm{Sink}(A) = RAC$:

$$\begin{aligned} s_{11} &= r_1 a_{11} c_1 & s_{12} &= r_1 a_{12} c_2 & s_{13} &= r_1 a_{13} c_3 \\ s_{21} &= r_2 a_{21} c_1 & s_{22} &= r_2 a_{22} c_2 & s_{23} &= r_2 a_{23} c_3 \\ s_{31} &= r_3 a_{31} c_1 & s_{32} &= r_3 a_{32} c_2 & s_{33} &= r_3 a_{33} c_3 \end{aligned}$$

6 equations from row and column sums:

$$\begin{aligned} s_{11} + s_{12} + s_{13} &= 1 & s_{11} + s_{21} + s_{31} &= 1 \\ s_{21} + s_{22} + s_{23} &= 1 & s_{12} + s_{22} + s_{32} &= 1 \\ s_{31} + s_{32} + s_{33} &= 1 & s_{13} + s_{23} + s_{33} &= 1 \end{aligned}$$

Want s_{11} in terms of a_{ij}.
15 equations; eliminate 14 variables $r_1, r_2, r_3, c_1, c_2, c_3, s_{12}, s_{13}, \ldots, s_{33}$.
Gröbner basis computation...

Theorem

The top-left entry $x = s_{11}$ *satisfies* $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, *where...*

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$$
\begin{aligned}
b_6 = {} & (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\
& \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\
b_5 = {} & -6a_{11}^5a_{22}^2a_{23}a_{32}a_{33}^2 + 6a_{11}^5a_{22}a_{23}^2a_{32}^2a_{33} + 8a_{11}^4a_{12}a_{21}a_{22}a_{23}a_{32}a_{33}^2 \\
& - 5a_{11}^4a_{12}a_{21}a_{23}^2a_{32}^2a_{33} + 5a_{11}^4a_{12}a_{22}^2a_{23}a_{31}a_{33}^2 - 8a_{11}^4a_{12}a_{22}a_{23}^2a_{31}a_{32}a_{33} \\
& + 5a_{11}^4a_{13}a_{21}a_{22}^2a_{32}a_{33}^2 - 8a_{11}^4a_{13}a_{21}a_{22}a_{23}a_{32}^2a_{33} + 8a_{11}^4a_{13}a_{22}^2a_{23}a_{31}a_{32}a_{33} \\
& - 5a_{11}^4a_{13}a_{22}a_{23}^2a_{31}a_{32}^2 - 2a_{11}^3a_{12}^2a_{21}^2a_{23}a_{32}a_{33}^2 - 6a_{11}^3a_{12}^2a_{21}a_{22}a_{23}a_{31}a_{33}^2 \\
& + 6a_{11}^3a_{12}^2a_{21}a_{23}^2a_{31}a_{32}a_{33} + 2a_{11}^3a_{12}^2a_{22}a_{23}^2a_{31}^2a_{33} - 6a_{11}^3a_{12}a_{13}a_{21}^2a_{22}a_{32}a_{33}^2 \\
& + 6a_{11}^3a_{12}a_{13}a_{21}^2a_{23}a_{32}^2a_{33} - 4a_{11}^3a_{12}a_{13}a_{21}a_{22}^2a_{31}a_{33}^2 + 4a_{11}^3a_{12}a_{13}a_{21}a_{23}^2a_{31}a_{32}^2 \\
& - 6a_{11}^3a_{12}a_{13}a_{22}^2a_{23}a_{31}^2a_{33} + 6a_{11}^3a_{12}a_{13}a_{22}a_{23}^2a_{31}^2a_{32} + 2a_{11}^3a_{13}^2a_{21}^2a_{22}a_{32}^2a_{33} \\
& - 6a_{11}^3a_{13}^2a_{21}a_{22}^2a_{31}a_{32}a_{33} + 6a_{11}^3a_{13}^2a_{21}a_{22}a_{23}a_{31}a_{32}^2 - 2a_{11}^3a_{13}^2a_{22}^2a_{23}a_{31}^2a_{32} \\
& + a_{11}^2a_{12}^3a_{21}^2a_{23}a_{31}a_{33}^2 - a_{11}^2a_{12}^3a_{21}a_{23}^2a_{31}^2a_{33} + a_{11}^2a_{12}^2a_{13}a_{21}^3a_{32}a_{33}^2 \\
& + 4a_{11}^2a_{12}^2a_{13}a_{21}^2a_{22}a_{31}a_{33}^2 - 4a_{11}^2a_{12}^2a_{13}a_{21}^2a_{23}a_{31}a_{32}a_{33} \\
& + 4a_{11}^2a_{12}^2a_{13}a_{21}a_{22}a_{23}a_{31}^2a_{33} - 4a_{11}^2a_{12}^2a_{13}a_{21}a_{23}^2a_{31}^2a_{32} - a_{11}^2a_{12}^2a_{13}a_{22}a_{23}^2a_{31}^3 \\
& - a_{11}^2a_{12}a_{13}^2a_{21}^3a_{32}^2a_{33} + 4a_{11}^2a_{12}a_{13}^2a_{21}^2a_{22}a_{31}a_{32}a_{33} - 4a_{11}^2a_{12}a_{13}^2a_{21}^2a_{23}a_{31}a_{32}^2 \\
& + 4a_{11}^2a_{12}a_{13}^2a_{21}a_{22}^2a_{31}^2a_{33} - 4a_{11}^2a_{12}a_{13}^2a_{21}a_{22}a_{23}a_{31}^2a_{32} + a_{11}^2a_{12}a_{13}^2a_{22}^2a_{23}a_{31}^3 \\
& - a_{11}^2a_{13}^3a_{21}^2a_{22}a_{31}a_{32}^2 + a_{11}^2a_{13}^3a_{21}a_{22}^2a_{31}^2a_{32} - 2a_{11}a_{12}^2a_{13}^2a_{21}^2a_{22}a_{31}^2a_{33} \\
& + 2a_{11}a_{12}^2a_{13}^2a_{21}^2a_{23}a_{31}^2a_{32} - a_{12}^3a_{13}^2a_{21}^3a_{31}^2a_{33} + a_{12}^3a_{13}^2a_{21}^2a_{23}a_{31}^3 \\
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\end{aligned}
$$

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$
\begin{aligned}
b_4 = a_{11} \big(&15 a_{11}^4 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^4 a_{22} a_{23}^2 a_{32}^2 a_{33} - 12 a_{11}^3 a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&+ 10 a_{11}^3 a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 10 a_{11}^3 a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 12 a_{11}^3 a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&- 10 a_{11}^3 a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 12 a_{11}^3 a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 12 a_{11}^3 a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&+ 10 a_{11}^3 a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{11}^2 a_{12}^2 a_{21}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11}^2 a_{12}^2 a_{21} a_{22} a_{23} a_{31} a_{33}^2 \\
&- 6 a_{11}^2 a_{12}^2 a_{21} a_{23}^2 a_{31} a_{32} a_{33} - a_{11}^2 a_{12}^2 a_{22} a_{23}^2 a_{31}^2 a_{33} + 6 a_{11}^2 a_{12} a_{13} a_{21}^2 a_{22} a_{32} a_{33}^2 \\
&- 6 a_{11}^2 a_{12} a_{13} a_{21}^2 a_{23} a_{32}^2 a_{33} + 6 a_{11}^2 a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - 6 a_{11}^2 a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 \\
&+ 6 a_{11}^2 a_{12} a_{13} a_{22}^2 a_{23} a_{31}^2 a_{33} - 6 a_{11}^2 a_{12} a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} - a_{11}^2 a_{13}^2 a_{21}^2 a_{22} a_{32}^2 a_{33} \\
&+ 6 a_{11}^2 a_{13}^2 a_{21} a_{22}^2 a_{31} a_{32} a_{33} - 6 a_{11}^2 a_{13}^2 a_{21} a_{22} a_{23} a_{31} a_{32}^2 + a_{11}^2 a_{13}^2 a_{22}^2 a_{23} a_{31}^2 a_{32} \\
&- 2 a_{11} a_{12}^2 a_{13} a_{21}^2 a_{22} a_{31} a_{33}^2 + 2 a_{11} a_{12}^2 a_{13} a_{21} a_{23}^2 a_{31}^2 a_{32} + 2 a_{11} a_{12} a_{13}^2 a_{21}^2 a_{23} a_{31} a_{32}^2 \\
&- 2 a_{11} a_{12} a_{13}^2 a_{21} a_{22}^2 a_{31}^2 a_{33} - 3 a_{12}^2 a_{13}^2 a_{21}^2 a_{22} a_{31}^2 a_{33} + 3 a_{12}^2 a_{13}^2 a_{21}^2 a_{23} a_{31}^2 a_{32} \big) \\
b_3 = 2 a_{11}^2 \big(&-10 a_{11}^3 a_{22}^2 a_{23} a_{32} a_{33}^2 + 10 a_{11}^3 a_{22} a_{23}^2 a_{32}^2 a_{33} + 4 a_{11}^2 a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&- 5 a_{11}^2 a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} + 5 a_{11}^2 a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 - 4 a_{11}^2 a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&+ 5 a_{11}^2 a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - 4 a_{11}^2 a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} + 4 a_{11}^2 a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&- 5 a_{11}^2 a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 - a_{11} a_{12}^2 a_{21} a_{22} a_{23} a_{31} a_{33}^2 + a_{11} a_{12}^2 a_{21} a_{23}^2 a_{31} a_{32} a_{33} \\
&- a_{11} a_{12} a_{13} a_{21}^2 a_{22} a_{32} a_{33}^2 + a_{11} a_{12} a_{13} a_{21}^2 a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 \\
&+ 2 a_{11} a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 - a_{11} a_{12} a_{13} a_{22}^2 a_{23} a_{31}^2 a_{33} + a_{11} a_{12} a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} \\
&- a_{11} a_{13}^2 a_{21} a_{22}^2 a_{31} a_{32} a_{33} + a_{11} a_{13}^2 a_{21} a_{22} a_{23} a_{31} a_{32}^2 + a_{12}^2 a_{13} a_{21}^2 a_{23} a_{31} a_{32} a_{33} \\
&- a_{12}^2 a_{13} a_{21} a_{22} a_{23} a_{31}^2 a_{33} - a_{12} a_{13}^2 a_{21}^2 a_{22} a_{31} a_{32} a_{33} + a_{12} a_{13}^2 a_{21} a_{22} a_{23} a_{31}^2 a_{32} \big)
\end{aligned}
$$

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 &= a_{11}^3 \big(15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&\quad + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&\quad - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&\quad + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2\big) \\
b_1 &= a_{11}^4 \big(-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2\big) \\
b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} \left(a_{22} a_{33} - a_{23} a_{32}\right)
\end{aligned}$$

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 &= a_{11}^3 (15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&\quad + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&\quad - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&\quad + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
b_1 &= a_{11}^4 (-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 = a_{11}^3 \big(& 15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&+ 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&- 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&+ 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 \big) \\
b_1 = a_{11}^4 \big(& -6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&+ a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 \big) \\
b_0 = a_{11}^5 & a_{22} a_{23} a_{32} a_{33} \left(a_{22} a_{33} - a_{23} a_{32}\right)
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 = {} & \left(a_{11} a_{22} - a_{12} a_{21}\right)\left(a_{11} a_{23} - a_{13} a_{21}\right)\left(a_{11} a_{32} - a_{12} a_{31}\right)\left(a_{11} a_{33} - a_{13} a_{31}\right) \\
& \cdot \left(a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}\right)
\end{aligned}$$

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$
\begin{aligned}
b_2 = a_{11}^3 \big(&15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&+ 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&- 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&+ 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2\big) \\
b_1 = a_{11}^4 \big(&-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&+ a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2\big) \\
b_0 = a_{11}^5 & a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}
$$

Better formulation?

$$
\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}
$$

b_6: product of 5 minors

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 = a_{11}^3 \big(&15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&+ 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&- 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&+ 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2\big) \\
b_1 = a_{11}^4 \big(&-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&+ a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2\big) \\
b_0 = a_{11}^5 &a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

b_6: product of 5 minors involving a_{11}

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 = a_{11}^3 \big(&15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&+ 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&- 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&+ 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2\big) \\
b_1 = a_{11}^4 \big(&-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&+ a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2\big) \\
b_0 = a_{11}^5 &a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

b_6: product of 5 minors involving a_{11} and 0 not.

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$
\begin{aligned}
b_2 &= a_{11}^3 \big(15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2\big) \\
b_1 &= a_{11}^4 \big(-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2\big) \\
b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}
$$

Better formulation?

$$
\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}
$$

b_6: product of 5 minors involving a_{11} and 0 not.

b_0: product of 0 minors involving a_{11} and 5 not (and a_{11}^5).

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 &= a_{11}^3 \left(15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2\right. \\
&\quad + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&\quad - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&\quad \left. + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2\right) \\
b_1 &= a_{11}^4 \left(-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33}\right. \\
&\quad \left. + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2\right) \\
b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} \left(a_{22} a_{33} - a_{23} a_{32}\right)
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

Multiply each b_k by a_{11}.

b_6: product of 5 minors involving a_{11} and 0 not.

b_0: product of 0 minors involving a_{11} and 5 not (and a_{11}^5).

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 = a_{11}^3 (&15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&+ 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&- 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&+ 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
b_1 = a_{11}^4 (&-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&+ a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
b_0 = a_{11}^5 &a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

Multiply each b_k by a_{11}.

$a_{11} b_6$: product of 6 minors involving a_{11} and 0 not.

b_0: product of 0 minors involving a_{11} and 5 not (and a_{11}^5).

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

Multiply each b_k by a_{11}.

$a_{11} b_6$: product of 6 minors involving a_{11} and 0 not.

$a_{11} b_0$: product of 0 minors involving a_{11} and 6 not (and a_{11}^6).

Theorem

The top-left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 = a_{11}^3 (&15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&+ 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&- 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&+ 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
b_1 = a_{11}^4 (&-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&+ a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
b_0 = a_{11}^5 & a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

Multiply each b_k by a_{11}.

$a_{11} b_6$: product of 6 minors involving a_{11} and 0 not.

$a_{11} b_0$: product of 0 minors involving a_{11} and 6 not (and a_{11}^6).

$a_{11} b_k$: products of k minors involving a_{11} and $6 - k$ not?

Theorem (Rowland–Wu 2025+)

The top-left entry x satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where

$$
\begin{aligned}
a_{11} b_6 &= \Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{smallmatrix}\right) \\
a_{11} b_5 &= -3\Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} \end{smallmatrix}\right) - \Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{2,3\} \end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix} \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{smallmatrix}\right) \\
a_{11} b_4 &= 4\Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{2\} & \{3\} \\ \{\} & \{2\} & \{3\} & \{2\} \end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2,3\} \end{smallmatrix}\right) - 3\Sigma\left(\begin{smallmatrix} \{2\} & \{2\} & \{3\} & \{3\} \\ \{2\} & \{3\} & \{2\} & \{3\} \end{smallmatrix}\right) \\
a_{11} b_3 &= -4\Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{2\} \\ \{\} & \{2\} & \{3\} \end{smallmatrix}\right) - 5\Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{3\} \\ \{\} & \{2\} & \{3\} \end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix} \{\} & \{2\} & \{2,3\} \\ \{\} & \{2\} & \{2,3\} \end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix} \{2\} & \{2\} & \{3\} \\ \{2\} & \{3\} & \{2\} \end{smallmatrix}\right) - \Sigma\left(\begin{smallmatrix} \{2\} & \{3\} & \{2,3\} \\ \{2\} & \{3\} & \{2,3\} \end{smallmatrix}\right) \\
a_{11} b_2 &= 4\Sigma\left(\begin{smallmatrix} \{\} & \{2\} \\ \{\} & \{2\} \end{smallmatrix}\right) - 3\Sigma\left(\begin{smallmatrix} \{\} & \{2,3\} \\ \{\} & \{2,3\} \end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix} \{2\} & \{3\} \\ \{2\} & \{3\} \end{smallmatrix}\right) \\
a_{11} b_1 &= -3\Sigma\left(\begin{smallmatrix} \{\} \\ \{\} \end{smallmatrix}\right) - \Sigma\left(\begin{smallmatrix} \{2\} \\ \{2\} \end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix} \{2,3\} \\ \{2,3\} \end{smallmatrix}\right) \\
a_{11} b_0 &= \Sigma\left(\right).
\end{aligned}
$$

$\Sigma(S)$ is a sum of products of $|S|$ minors involving a_{11} and $6 - |S|$ not.
The pairs $\begin{smallmatrix} R \\ C \end{smallmatrix}$ specify row and column indices for the minors involving a_{11}.

Degree: number of minors not involving the first row or first column.

For $n \times n$ matrices: $\displaystyle\sum_{j=0}^{n-1} \binom{n-1}{j}^2 = \binom{2n-2}{n-1}$ $\qquad 1, 2, 6, 20, 70, 252, \ldots$

What are the integer coefficients?

Gröbner basis computations are infeasible for $n \geq 4$.

Instead, use PSLQ on examples and interpolate.
Infeasible for $n \geq 6$.

Definition

Let A be an $m \times n$ matrix with positive entries.
The *Sinkhorn limit* of A is obtained by iteratively scaling so that each row sum is 1 and each column sum is $\frac{m}{n}$.

The limit exists: Sinkhorn 1967 (in a more general form).

1.5 CPU years scaling matrices and recognizing 102K algebraic numbers let us solve for 63K coefficients (and 56K parameterized by free variables).

Conjecture (Rowland–Wu 2025+)

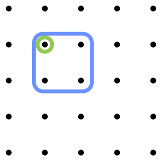
The coefficient of $\Sigma(S) x^{|S|}$ *is* $\det\left(\frac{1}{m} \operatorname{adj}_S(m,n)\right)$.

In particular, the top-left entry has degree at most $\binom{m+n-2}{m-1}$.
This bound was proved in May 2025 by Fang.

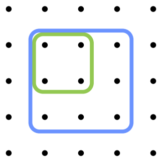
$\operatorname{adj}_S(m,n)$ is an adjacency-like matrix.
Underlying graph: Vertex set S. Edges?

Type-1 edges: Sizes differ by 1, and one is a subset of the other.

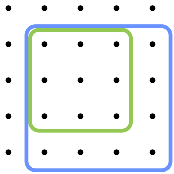
$$S = \begin{matrix} \{2\} & \{2,3\} \\ \{2\} & \{2,3\} \end{matrix}$$



$$S = \begin{matrix} \{2,3\} & \{2,3,4\} \\ \{2,3\} & \{2,3,4\} \end{matrix}$$

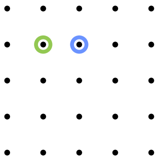


$$S = \begin{matrix} \{2,3,4\} & \{2,3,4,5\} \\ \{2,3,4\} & \{2,3,4,5\} \end{matrix}$$

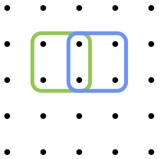


Type-2 edges: Same sizes, and they differ in exactly 1 row or 1 column.

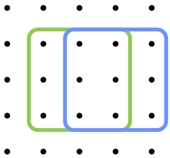
$$S = \begin{matrix} \{2\} & \{2\} \\ \{2\} & \{3\} \end{matrix}$$



$$S = \begin{matrix} \{2,3\} & \{2,3\} \\ \{2,3\} & \{3,4\} \end{matrix}$$



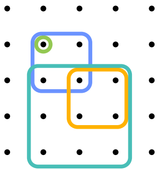
$$S = \begin{matrix} \{2,3,4\} & \{2,3,4\} \\ \{2,3,4\} & \{3,4,5\} \end{matrix}$$



Connected components are built from these.

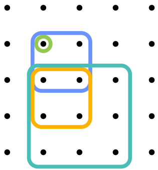
Two components:

$$S = \begin{matrix} \{2\} & \{2,3\} & \{3,4\} & \{3,4,5\} \\ \{2\} & \{2,3\} & \{3,4\} & \{2,3,4\} \end{matrix}$$



One component:

$$S = \begin{matrix} \{2\} & \{2,3\} & \{3,4\} & \{3,4,5\} \\ \{2\} & \{2,3\} & \{2,3\} & \{2,3,4\} \end{matrix}$$



References

- Shalosh B. Ekhad and Doron Zeilberger, Answers to some questions about explicit Sinkhorn limits posed by Mel Nathanson, `https://arxiv.org/abs/1902.10783` (6 pages).
- Max Chicky Fang, Closed form of a generalized Sinkhorn limit, `https://arxiv.org/abs/2506.06338` (17 pages).
- Melvyn B. Nathanson, Alternate minimization and doubly stochastic matrices, *Integers* **20A** (2020) Article #A10 (17 pages).
- Eric Rowland and Jason Wu, The entries of the Sinkhorn limit of an $m \times n$ matrix, `https://arxiv.org/abs/2409.02789` (27 pages).
- Richard Sinkhorn, A relationship between arbitrary positive matrices and doubly stochastic matrices, *The Annals of Mathematical Statistics* **35** (1964) 876–879.