Combinatorial structure behind Sinkhorn limits

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Problem

Given a square matrix with positive entries, turn it into a "close" doubly stochastic matrix of the same size (row and column sums are 1).

2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

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Scale rows . . .

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$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows . . .

$$\begin{bmatrix} .800000 & .200000 \\ .666667 & .333333 \end{bmatrix}$$

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2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns ...

$$\begin{bmatrix} .545455 & .375000 \\ .454545 & .625000 \end{bmatrix}$$

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2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows . . .

$$\begin{bmatrix} .592593 & .407407 \\ .421053 & .578947 \end{bmatrix}$$

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2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .584615 & .413043 \\ .415385 & .586957 \end{bmatrix}$$

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$$\begin{bmatrix} .585987 & .414013 \\ .414414 & .585586 \end{bmatrix}$$

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$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585752 & .414179 \\ .414248 & .585821 \end{bmatrix}$$

Problem

Given a square matrix with positive entries, turn it into a "close" doubly stochastic matrix of the same size (row and column sums are 1).

2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585792 & .414208 \\ .414219 & .585781 \end{bmatrix}$$

Problem

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$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585785 & .414213 \\ .414215 & .585787 \end{bmatrix}$$

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In the limit, we obtain the Sinkhorn limit of $\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$.

Sinkhorn 1964: The limit exists.

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Given a square matrix with positive entries, turn it into a "close" doubly stochastic matrix of the same size (row and column sums are 1).

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$$\begin{bmatrix} .585786 & .414214 \\ .414214 & .585786 \end{bmatrix} \approx \begin{bmatrix} 2-\sqrt{2} & -1+\sqrt{2} \\ -1+\sqrt{2} & 2-\sqrt{2} \end{bmatrix}$$

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Applications in computer science:

- preconditioning a linear system to improve numerical stability
- approximating the permanent of a matrix
- determining whether a graph has a perfect matching

Applications in other areas:

- predicting telephone traffic (Kruithof 1937)
  video: https://www.youtube.com/@EricRowland
- transportation science (Deming–Stephan 1940)
- economics (Stone 1964)
- image processing (Herman–Lent 1976)
- operations research (Raghavan 1984)
- machine learning (Cuturi 2013)

Idel (2016) wrote a 100-page survey of Sinkhorn-related results.

Question

What are the exact entries of the Sinkhorn limit?

Notation:

$$\text{Sink}\left(\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2-\sqrt{2} & -1+\sqrt{2} \\ -1+\sqrt{2} & 2-\sqrt{2} \end{bmatrix}$$

Theorem (Nathanson 2020)

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with positive entries,

$$\text{Sink}(A) = \frac{1}{\sqrt{ad}+\sqrt{bc}} \begin{bmatrix} \sqrt{ad} & \sqrt{bc} \\ \sqrt{bc} & \sqrt{ad} \end{bmatrix}.$$

For a symmetric 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ with positive entries:

Theorem (Ekhad–Zeilberger 2019)

The top left entry x of $\mathrm{Sink}(A)$ *satisfies* $c_4 x^4 + \cdots + c_1 x + c_0 = 0$, *where*

$$c_4 = -(a_{12}^2 - a_{11}a_{22})(a_{13}^2 - a_{11}a_{33})(-a_{11}a_{22}a_{33} + a_{11}a_{23}^2 + a_{12}^2 a_{33} - 2a_{12}a_{13}a_{23} + a_{13}^2 a_{22})$$

$$c_3 = (-4a_{11}^3 a_{22}^2 a_{33}^2 + 4a_{11}^3 a_{22} a_{23}^2 a_{33} + 4a_{11}^2 a_{12}^2 a_{22} a_{33}^2 - 3a_{11}^2 a_{12}^2 a_{23}^2 a_{33} - 2a_{11}^2 a_{12} a_{13} a_{22} a_{23} a_{33} + 4a_{11}^2 a_{13}^2 a_{22}^2 a_{33} \\ - 3a_{11}^2 a_{13}^2 a_{22} a_{23}^2 - 2a_{11} a_{12}^2 a_{13}^2 a_{22} a_{33} + 2a_{11} a_{12}^2 a_{13}^2 a_{23}^2 - a_{12}^4 a_{13}^2 a_{33} + 2a_{12}^3 a_{13}^3 a_{23} - a_{12}^2 a_{13}^4 a_{22})$$

$$c_2 = a_{11}(6a_{11}^2 a_{22}^2 a_{33}^2 - 6a_{11}^2 a_{22} a_{23}^2 a_{33} - 2a_{11} a_{12}^2 a_{22} a_{33}^2 + 3a_{11} a_{12}^2 a_{23}^2 a_{33} - 2a_{11} a_{12} a_{13} a_{22} a_{23} a_{33} - 2a_{11} a_{13}^2 a_{22}^2 a_{33} \\ + 3a_{11} a_{13}^2 a_{22} a_{23}^2 + 2a_{12}^3 a_{13} a_{23} a_{33} - 3a_{12}^2 a_{13}^2 a_{22} a_{33} - a_{12}^2 a_{13}^2 a_{23}^2 + 2a_{12} a_{13}^3 a_{22} a_{23})$$

$$c_1 = -a_{11}^2(4a_{11} a_{22}^2 a_{33}^2 - 4a_{11} a_{22} a_{23}^2 a_{33} + a_{12}^2 a_{23}^2 a_{33} - 2a_{12} a_{13} a_{22} a_{23} a_{33} + a_{13}^2 a_{22} a_{23}^2)$$

$$c_0 = a_{11}^3 a_{22} a_{33}(a_{22} a_{33} - a_{23}^2)$$

Computed with Gröbner bases.

The entries are algebraic with degree at most 4.

For general 3×3 matrices, the Sinkhorn limit wasn't known!

$$\operatorname{Sink}\left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix}\right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

Can we identify these numbers?

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Compute the top left entry to high precision:
$x \approx .2503383740593684894545472868514292528338672217353016771994$

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Can we identify these numbers?

Compute the top left entry to high precision:
$x \approx .25033837405936848945454728685142925283386722173530167719 94$

Assume it's algebraic. Guess the degree:

For general 3×3 matrices, the Sinkhorn limit wasn't known!

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Compute the top left entry to high precision:
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Assume it's algebraic. Guess the degree: 4
Use the PSLQ integer relation algorithm to find a likely polynomial:

$$b_4 x^4 + b_3 x^3 + b_2 x^2 + b_1 x + b_0 \approx 0$$

For general 3×3 matrices, the Sinkhorn limit wasn't known!

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Assume it's algebraic. Guess the degree: 4
Use the PSLQ integer relation algorithm to find a likely polynomial:

$$36164989943x^4 + 333428071444x^3 + 65054452280x^2 - 41075578985x + 832844043 \approx 0$$

For general 3×3 matrices, the Sinkhorn limit wasn't known!

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garbage!

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Can we identify these numbers?

Compute the top left entry to high precision:
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Assume it's algebraic. Guess the degree: 5
Use the PSLQ integer relation algorithm to find a likely polynomial:

$$860608662x^5 - 311838369x^4 + 602643248x^3 + \cdots - 1242712455x + 300075073 \approx 0$$

For general 3×3 matrices, the Sinkhorn limit wasn't known!

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Can we identify these numbers?

Compute the top left entry to high precision:
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Can we identify these numbers?

Compute the top left entry to high precision:
$x \approx .25033837405936848945454728685142925283386722173530167719994$

Assume it's algebraic. Guess the degree: 6
Use the PSLQ integer relation algorithm to find a likely polynomial:

$$236379x^6 + 502124x^5 - 1610856x^4 + 19808x^3 + 661120x^2 - 94592x - 12288 = 0$$

For general 3×3 matrices, the Sinkhorn limit wasn't known!

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Conjecture (Chen and Varghese 2019, Hofstra SSRP)

For 3×3 matrices A, the entries of $\operatorname{Sink}(A)$ have degree at most 6.

It suffices to describe the top left entry of $\mathrm{Sink}(A)$.

Fact

If we know one entry of $\mathrm{Sink}(A)$ as a function of A, then we know them all.

Reason: Iterative scaling isn't sensitive to row or column order.

For a 3×3 matrix, what is the top left entry of $\mathrm{Sink}(A)$? System of equations...

Row scaling — multiplication on the left.
Column scaling — multiplication on the right.

$$\mathrm{Sink}(A) = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \qquad R = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad C = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

9 equations from $\mathrm{Sink}(A) = RAC$:

$$\begin{aligned} s_{11} &= r_1 a_{11} c_1 & s_{12} &= r_1 a_{12} c_2 & s_{13} &= r_1 a_{13} c_3 \\ s_{21} &= r_2 a_{21} c_1 & s_{22} &= r_2 a_{22} c_2 & s_{23} &= r_2 a_{23} c_3 \\ s_{31} &= r_3 a_{31} c_1 & s_{32} &= r_3 a_{32} c_2 & s_{33} &= r_3 a_{33} c_3 \end{aligned}$$

6 equations from row and column sums:

$$\begin{aligned} s_{11} + s_{12} + s_{13} &= 1 & s_{11} + s_{21} + s_{31} &= 1 \\ s_{21} + s_{22} + s_{23} &= 1 & s_{12} + s_{22} + s_{32} &= 1 \\ s_{31} + s_{32} + s_{33} &= 1 & s_{13} + s_{23} + s_{33} &= 1 \end{aligned}$$

Want s_{11} in terms of a_{ij}.
15 equations; eliminate 14 variables $r_1, r_2, r_3, c_1, c_2, c_3, s_{12}, s_{13}, \dots, s_{33}$.
Gröbner basis computation...

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$
\begin{aligned}
b_6 = {} & (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\
& \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\
b_5 = {} & -6a_{11}^5a_{22}^2a_{23}a_{32}a_{33}^2 + 6a_{11}^5a_{22}a_{23}^2a_{32}^2a_{33} + 8a_{11}^4a_{12}a_{21}a_{22}a_{23}a_{32}a_{33}^2 \\
& - 5a_{11}^4a_{12}a_{21}a_{23}^2a_{32}^2a_{33} + 5a_{11}^4a_{12}a_{22}^2a_{23}a_{31}a_{33}^2 - 8a_{11}^4a_{12}a_{22}a_{23}^2a_{31}a_{32}a_{33} \\
& + 5a_{11}^4a_{13}a_{21}a_{22}^2a_{32}a_{33}^2 - 8a_{11}^4a_{13}a_{21}a_{22}a_{23}a_{32}^2a_{33} + 8a_{11}^4a_{13}a_{22}^2a_{23}a_{31}a_{32}a_{33} \\
& - 5a_{11}^4a_{13}a_{22}a_{23}^2a_{31}a_{32}^2 - 2a_{11}^3a_{12}^2a_{21}^2a_{23}a_{32}a_{33}^2 - 6a_{11}^3a_{12}^2a_{21}a_{22}a_{23}a_{31}a_{33}^2 \\
& + 6a_{11}^3a_{12}^2a_{21}a_{23}^2a_{31}a_{32}a_{33} + 2a_{11}^3a_{12}^2a_{22}a_{23}^2a_{31}^2a_{33} - 6a_{11}^3a_{12}a_{13}a_{21}^2a_{22}a_{32}a_{33}^2 \\
& + 6a_{11}^3a_{12}a_{13}a_{21}^2a_{23}a_{32}^2a_{33} - 4a_{11}^3a_{12}a_{13}a_{21}a_{22}^2a_{31}a_{33}^2 + 4a_{11}^3a_{12}a_{13}a_{21}a_{23}^2a_{31}a_{32}^2 \\
& - 6a_{11}^3a_{12}a_{13}a_{22}^2a_{23}a_{31}^2a_{33} + 6a_{11}^3a_{12}a_{13}a_{22}a_{23}^2a_{31}^2a_{32} + 2a_{11}^3a_{13}^2a_{21}^2a_{22}a_{32}^2a_{33} \\
& - 6a_{11}^3a_{13}^2a_{21}a_{22}^2a_{31}a_{32}a_{33} + 6a_{11}^3a_{13}^2a_{21}a_{22}a_{23}a_{31}a_{32}^2 - 2a_{11}^3a_{13}^2a_{22}^2a_{23}a_{31}^2a_{32} \\
& + a_{11}^2a_{12}^3a_{21}^2a_{23}a_{31}a_{33}^2 - a_{11}^2a_{12}^3a_{21}a_{23}^2a_{31}^2a_{33} + a_{11}^2a_{12}^2a_{13}a_{21}^3a_{32}a_{33}^2 \\
& + 4a_{11}^2a_{12}^2a_{13}a_{21}^2a_{22}a_{31}a_{33}^2 - 4a_{11}^2a_{12}^2a_{13}a_{21}^2a_{23}a_{31}a_{32}a_{33} \\
& + 4a_{11}^2a_{12}^2a_{13}a_{21}a_{22}a_{23}a_{31}^2a_{33} - 4a_{11}^2a_{12}^2a_{13}a_{21}a_{23}^2a_{31}^2a_{32} - a_{11}^2a_{12}^2a_{13}a_{22}a_{23}^2a_{31}^3 \\
& - a_{11}^2a_{12}a_{13}^2a_{21}^3a_{32}^2a_{33} + 4a_{11}^2a_{12}a_{13}^2a_{21}^2a_{22}a_{31}a_{32}a_{33} - 4a_{11}^2a_{12}a_{13}^2a_{21}^2a_{23}a_{31}a_{32}^2 \\
& + 4a_{11}^2a_{12}a_{13}^2a_{21}a_{22}^2a_{31}^2a_{33} - 4a_{11}^2a_{12}a_{13}^2a_{21}a_{22}a_{23}a_{31}^2a_{32} + a_{11}^2a_{12}a_{13}^2a_{22}^2a_{23}a_{31}^3 \\
& - a_{11}^2a_{13}^3a_{21}^2a_{22}a_{31}a_{32}^2 + a_{11}^2a_{13}^3a_{21}a_{22}^2a_{31}^2a_{32} - 2a_{11}a_{12}^2a_{13}^2a_{21}^2a_{22}a_{31}^2a_{33} \\
& + 2a_{11}a_{12}^2a_{13}^2a_{21}^2a_{23}a_{31}^2a_{32} - a_{12}^3a_{13}^2a_{21}^3a_{31}^2a_{33} + a_{12}^3a_{13}^2a_{21}^2a_{23}a_{31}^3 \\
& + a_{12}^2a_{13}^3a_{21}^3a_{31}^2a_{32} - a_{12}^2a_{13}^3a_{21}^2a_{22}a_{31}^3
\end{aligned}
$$

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_4 = a_{11}\big(&15a_{11}^4 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^4 a_{22} a_{23}^2 a_{32}^2 a_{33} - 12a_{11}^3 a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&+ 10a_{11}^3 a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 10a_{11}^3 a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 12a_{11}^3 a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&- 10a_{11}^3 a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 12a_{11}^3 a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 12a_{11}^3 a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&+ 10a_{11}^3 a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{11}^2 a_{12}^2 a_{21}^2 a_{23} a_{32} a_{33}^2 + 6a_{11}^2 a_{12}^2 a_{21} a_{22} a_{23} a_{31} a_{33}^2 \\
&- 6a_{11}^2 a_{12}^2 a_{21} a_{23}^2 a_{31} a_{32} a_{33} - a_{11}^2 a_{12}^2 a_{22} a_{23}^2 a_{31}^2 a_{33} + 6a_{11}^2 a_{12} a_{13} a_{21}^2 a_{22} a_{32} a_{33}^2 \\
&- 6a_{11}^2 a_{12} a_{13} a_{21}^2 a_{23} a_{32}^2 a_{33} + 6a_{11}^2 a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - 6a_{11}^2 a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 \\
&+ 6a_{11}^2 a_{12} a_{13} a_{22}^2 a_{23} a_{31}^2 a_{33} - 6a_{11}^2 a_{12} a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} - a_{11}^2 a_{13}^2 a_{21}^2 a_{22} a_{32}^2 a_{33} \\
&+ 6a_{11}^2 a_{13}^2 a_{21} a_{22}^2 a_{31} a_{32} a_{33} - 6a_{11}^2 a_{13}^2 a_{21} a_{22} a_{23} a_{31} a_{32}^2 + a_{11}^2 a_{13}^2 a_{22}^2 a_{23} a_{31}^2 a_{32} \\
&- 2a_{11} a_{12}^2 a_{13} a_{21}^2 a_{22} a_{31} a_{33}^2 + 2a_{11} a_{12}^2 a_{13} a_{21} a_{23}^2 a_{31}^2 a_{32} + 2a_{11} a_{12} a_{13}^2 a_{21}^2 a_{23} a_{31} a_{32}^2 \\
&- 2a_{11} a_{12} a_{13}^2 a_{21} a_{22}^2 a_{31}^2 a_{33} - 3a_{12}^2 a_{13}^2 a_{21}^2 a_{22} a_{31}^2 a_{33} + 3a_{12}^2 a_{13}^2 a_{21}^2 a_{23} a_{31}^2 a_{32}\big) \\
b_3 = 2a_{11}^2\big(&-10a_{11}^3 a_{22}^2 a_{23} a_{32} a_{33}^2 + 10a_{11}^3 a_{22} a_{23}^2 a_{32}^2 a_{33} + 4a_{11}^2 a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&- 5a_{11}^2 a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} + 5a_{11}^2 a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 - 4a_{11}^2 a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&+ 5a_{11}^2 a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - 4a_{11}^2 a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} + 4a_{11}^2 a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&- 5a_{11}^2 a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 - a_{11} a_{12}^2 a_{21} a_{22} a_{23} a_{31} a_{33}^2 + a_{11} a_{12}^2 a_{21} a_{23}^2 a_{31} a_{32} a_{33} \\
&- a_{11} a_{12} a_{13} a_{21}^2 a_{22} a_{32} a_{33}^2 + a_{11} a_{12} a_{13} a_{21}^2 a_{23} a_{32}^2 a_{33} - 2a_{11} a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 \\
&+ 2a_{11} a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 - a_{11} a_{12} a_{13} a_{22}^2 a_{23} a_{31}^2 a_{33} + a_{11} a_{12} a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} \\
&- a_{11} a_{13}^2 a_{21} a_{22}^2 a_{31} a_{32} a_{33} + a_{11} a_{13}^2 a_{21} a_{22} a_{23} a_{31} a_{32}^2 + a_{12}^2 a_{13} a_{21}^2 a_{23} a_{31} a_{32} a_{33} \\
&- a_{12}^2 a_{13} a_{21} a_{22} a_{23} a_{31}^2 a_{33} - a_{12} a_{13}^2 a_{21}^2 a_{22} a_{31} a_{32} a_{33} + a_{12} a_{13}^2 a_{21} a_{22} a_{23} a_{31}^2 a_{32}\big)
\end{aligned}$$

Theorem

The top left entry $x = s_{11}$ *satisfies* $b_6 x^6 + \cdots + b_1 x + b_0 = 0$*, where...*

$$\begin{aligned} b_2 &= a_{11}^3 (15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\ &\quad + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\ &\quad - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\ &\quad + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\ b_1 &= a_{11}^4 (-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\ &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\ b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32}) \end{aligned}$$

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 &= a_{11}^3 \big(15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2\big) \\
b_1 &= a_{11}^4 \big(-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2\big) \\
b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} \left(a_{22} a_{33} - a_{23} a_{32}\right)
\end{aligned}$$

Better formulation?

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 = {} & a_{11}^3 \big(15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
& + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
& - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
& + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2\big) \\
b_1 = {} & a_{11}^4 \big(-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
& + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2\big) \\
b_0 = {} & a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 = {} & (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
& \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31})
\end{aligned}$$

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 = a_{11}^3 (&15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&+ 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&- 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&+ 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
b_1 = a_{11}^4 (&-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&+ a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
b_0 = a_{11}^5 &a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

b_6: product of 5 minors

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 = {} & a_{11}^3 (15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
& + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
& - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
& + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
b_1 = {} & a_{11}^4 (-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
& + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
b_0 = {} & a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 = {} & (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
& \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
= {} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

b_6: product of 5 minors involving a_{11}

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 &= a_{11}^3 \left(15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \right. \\
&\quad + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&\quad - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&\quad \left. + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 \right) \\
b_1 &= a_{11}^4 \left(-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \right. \\
&\quad \left. + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 \right) \\
b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} \left(a_{22} a_{33} - a_{23} a_{32}\right)
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

b_6: product of 5 minors involving a_{11} and 0 not.

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned} b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\ &\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\ &\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\ &\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\ b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\ &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\ b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32}) \end{aligned}$$

Better formulation?

$$\begin{aligned} b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\ &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{aligned}$$

b_6: product of 5 minors involving a_{11} and 0 not.

b_0: product of 0 minors involving a_{11} and 5 not (and a_{11}^5).

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 &= a_{11}^3 \left(15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \right.\\
&\quad + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33}\\
&\quad - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33}\\
&\quad \left. + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2\right)\\
b_1 &= a_{11}^4 \left(-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \right.\\
&\quad \left. + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2\right)\\
b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31})\\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31})\\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

Multiply each b_k by a_{11}.

- b_6: product of 5 minors involving a_{11} and 0 not.
- b_0: product of 0 minors involving a_{11} and 5 not (and a_{11}^5).

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 = {} & a_{11}^3 \big(15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
& + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
& - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
& + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2\big) \\
b_1 = {} & a_{11}^4 \big(-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
& + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2\big) \\
b_0 = {} & a_{11}^5 a_{22} a_{23} a_{32} a_{33} \left(a_{22} a_{33} - a_{23} a_{32}\right)
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

Multiply each b_k by a_{11}.

$a_{11} b_6$: product of 6 minors involving a_{11} and 0 not.

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Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

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b_2 = a_{11}^3 \big(&15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&+ 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&- 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&+ 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2\big) \\
b_1 = a_{11}^4 \big(&-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&+ a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2\big) \\
b_0 = a_{11}^5 &a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

Multiply each b_k by a_{11}.

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$a_{11} b_0$: product of 0 minors involving a_{11} and 6 not (and a_{11}^6).

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 = {} & a_{11}^3 \left(15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2\right. \\
& + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
& - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
& \left. + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2\right) \\
b_1 = {} & a_{11}^4 \left(-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33}\right. \\
& \left. + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2\right) \\
b_0 = {} & a_{11}^5 a_{22} a_{23} a_{32} a_{33} \left(a_{22} a_{33} - a_{23} a_{32}\right)
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 = {} & (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
& \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
= {} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

Multiply each b_k by a_{11}.

$a_{11} b_6$: product of 6 minors involving a_{11} and 0 not.

$a_{11} b_0$: product of 0 minors involving a_{11} and 6 not (and a_{11}^6).

$a_{11} b_k$: products of k minors involving a_{11} and $6 - k$ not?

Theorem (Rowland–Wu 2025)

Let A be a 3×3 matrix with positive entries.
The top left entry x of $\mathrm{Sink}(A)$ *satisfies* $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, *where*

$$a_{11} b_6 = \Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}\ \{3\}\ \{2,3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}\ \{3\}\ \{2,3\}}$$

$$a_{11} b_5 = -3\Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}\ \{3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}\ \{3\}} - \Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}\ \{2,3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}\ \{2,3\}} + \Sigma\binom{\{2\}\ \{2\}\ \{3\}\ \{3\}\ \{2,3\}}{\{2\}\ \{3\}\ \{2\}\ \{3\}\ \{2,3\}}$$

$$a_{11} b_4 = 4\Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}} + \Sigma\binom{\{\}\ \{2\}\ \{3\}\ \{2,3\}}{\{\}\ \{2\}\ \{3\}\ \{2,3\}} - 3\Sigma\binom{\{2\}\ \{2\}\ \{3\}\ \{3\}}{\{2\}\ \{3\}\ \{2\}\ \{3\}}$$

$$a_{11} b_3 = -4\Sigma\binom{\{\}\ \{2\}\ \{2\}}{\{\}\ \{2\}\ \{3\}} - 5\Sigma\binom{\{\}\ \{2\}\ \{3\}}{\{\}\ \{2\}\ \{3\}} + \Sigma\binom{\{\}\ \{2\}\ \{2,3\}}{\{\}\ \{2\}\ \{2,3\}} + \Sigma\binom{\{2\}\ \{2\}\ \{3\}}{\{2\}\ \{3\}\ \{2\}} - \Sigma\binom{\{2\}\ \{3\}\ \{2,3\}}{\{2\}\ \{3\}\ \{2,3\}}$$

$$a_{11} b_2 = 4\Sigma\binom{\{\}\ \{2\}}{\{\}\ \{2\}} - 3\Sigma\binom{\{\}\ \{2,3\}}{\{\}\ \{2,3\}} + \Sigma\binom{\{2\}\ \{3\}}{\{2\}\ \{3\}}$$

$$a_{11} b_1 = -3\Sigma\binom{\{\}}{\{\}} - \Sigma\binom{\{2\}}{\{2\}} + \Sigma\binom{\{2,3\}}{\{2,3\}}$$

$$a_{11} b_0 = \Sigma\left(\right).$$

$\Sigma(S)$ is a sum of products of $|S|$ minors involving a_{11} and $6 - |S|$ not.

The pairs $\frac{R}{C}$ specify row and column indices for the minors involving a_{11}.

Formal definitions:

Let $R \subseteq \{2,3\}$ and $C \subseteq \{2,3\}$ with $|R| = |C|$.

$$\Delta\binom{R}{C} = \det A_{\{1\}\cup R, \{1\}\cup C} \qquad M(S) = \prod_{(R,C)\in S} \Delta\binom{R}{C} \cdot \prod_{(R,C)\notin S} \Gamma\binom{R}{C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C} \qquad \Sigma(S) = \sum_{T\equiv S} M(T)$$

Coefficients:

$$a_{11} b_6 = a_{11}\left(a_{11}a_{22} - a_{12}a_{21}\right)\left(a_{11}a_{23} - a_{13}a_{21}\right)\left(a_{11}a_{32} - a_{12}a_{31}\right)\left(a_{11}a_{33} - a_{13}a_{31}\right)$$
$$\cdot \left(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}\right)$$

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Coefficients:

$$\begin{aligned} a_{11} b_6 &= a_{11} (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= \Delta\binom{\{\}}{\{\}} \Delta\binom{\{2\}}{\{2\}} \Delta\binom{\{2\}}{\{3\}} \Delta\binom{\{3\}}{\{2\}} \Delta\binom{\{3\}}{\{3\}} \Delta\binom{\{2,3\}}{\{2,3\}} \end{aligned}$$

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$$a_{11} b_0 = a_{11}^6 a_{22}a_{23}a_{32}a_{33}\,(a_{22}a_{33} - a_{23}a_{32})$$

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&\qquad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\
&= \Delta\binom{\{\}}{\{\}}\,\Delta\binom{\{2\}}{\{2\}}\,\Delta\binom{\{2\}}{\{3\}}\,\Delta\binom{\{3\}}{\{2\}}\,\Delta\binom{\{3\}}{\{3\}}\,\Delta\binom{\{2,3\}}{\{2,3\}} \\
&= M\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
a_{11} b_0 &= a_{11}^6 a_{22}a_{23}a_{32}a_{33}\,(a_{22}a_{33} - a_{23}a_{32}) \\
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Coefficients:

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&\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\
&= \Delta\binom{\{\}}{\{\}} \Delta\binom{\{2\}}{\{2\}} \Delta\binom{\{2\}}{\{3\}} \Delta\binom{\{3\}}{\{2\}} \Delta\binom{\{3\}}{\{3\}} \Delta\binom{\{2,3\}}{\{2,3\}} \\
&= M\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
a_{11} b_0 &= a_{11}^6 a_{22}a_{23}a_{32}a_{33} (a_{22}a_{33} - a_{23}a_{32}) \\
&= \Gamma\binom{\{\}}{\{\}} \Gamma\binom{\{2\}}{\{2\}} \Gamma\binom{\{2\}}{\{3\}} \Gamma\binom{\{3\}}{\{2\}} \Gamma\binom{\{3\}}{\{3\}} \Gamma\binom{\{2,3\}}{\{2,3\}} \\
&= M\left(\right)
\end{aligned}$$

Formal definitions:
Let $R \subseteq \{2,3\}$ and $C \subseteq \{2,3\}$ with $|R| = |C|$.

$$\Delta\binom{R}{C} = \det A_{\{1\}\cup R, \{1\}\cup C} \qquad M(S) = \prod_{(R,C)\in S} \Delta\binom{R}{C} \cdot \prod_{(R,C)\notin S} \Gamma\binom{R}{C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C} \qquad \Sigma(S) = \sum_{T\equiv S} M(T)$$

Coefficients:

$$\begin{aligned}
a_{11} b_6 &= a_{11} (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \Delta\binom{\{\}}{\{\}} \Delta\binom{\{2\}}{\{2\}} \Delta\binom{\{2\}}{\{3\}} \Delta\binom{\{3\}}{\{2\}} \Delta\binom{\{3\}}{\{3\}} \Delta\binom{\{2,3\}}{\{2,3\}} \\
&= M\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
a_{11} b_0 &= a_{11}^6 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32}) \\
&= \Gamma\binom{\{\}}{\{\}} \Gamma\binom{\{2\}}{\{2\}} \Gamma\binom{\{2\}}{\{3\}} \Gamma\binom{\{3\}}{\{2\}} \Gamma\binom{\{3\}}{\{3\}} \Gamma\binom{\{2,3\}}{\{2,3\}} \\
&= M\Big(\Big)
\end{aligned}$$

$$\begin{aligned}
a_{11} b_1 &= a_{11}^5 (-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2)
\end{aligned}$$

Formal definitions:
Let $R \subseteq \{2,3\}$ and $C \subseteq \{2,3\}$ with $|R| = |C|$.

$$\Delta\binom{R}{C} = \det A_{\{1\}\cup R,\{1\}\cup C} \qquad M(S) = \prod_{(R,C)\in S} \Delta\binom{R}{C} \cdot \prod_{(R,C)\notin S} \Gamma\binom{R}{C}$$

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&\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\
&= \Delta\binom{\{\}}{\{\}}\,\Delta\binom{\{2\}}{\{2\}}\,\Delta\binom{\{2\}}{\{3\}}\,\Delta\binom{\{3\}}{\{2\}}\,\Delta\binom{\{3\}}{\{3\}}\,\Delta\binom{\{2,3\}}{\{2,3\}} \\
&= M\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
a_{11} b_0 &= a_{11}^6 a_{22}a_{23}a_{32}a_{33}\,(a_{22}a_{33} - a_{23}a_{32}) \\
&= \Gamma\binom{\{\}}{\{\}}\,\Gamma\binom{\{2\}}{\{2\}}\,\Gamma\binom{\{2\}}{\{3\}}\,\Gamma\binom{\{3\}}{\{2\}}\,\Gamma\binom{\{3\}}{\{3\}}\,\Gamma\binom{\{2,3\}}{\{2,3\}} \\
&= M\big(\big)
\end{aligned}$$

$$\begin{aligned}
a_{11} b_1 &= a_{11}^5\,(-6a_{11}a_{22}^2a_{23}a_{32}a_{33}^2 + 6a_{11}a_{22}a_{23}^2a_{32}^2a_{33} - a_{12}a_{21}a_{23}^2a_{32}^2a_{33} \\
&\quad + a_{12}a_{22}^2a_{23}a_{31}a_{33}^2 + a_{13}a_{21}a_{22}^2a_{32}a_{33}^2 - a_{13}a_{22}a_{23}^2a_{31}a_{32}^2) \\
&= -3M\binom{\{\}}{\{\}} - M\binom{\{2\}}{\{2\}} - M\binom{\{2\}}{\{3\}} - M\binom{\{3\}}{\{2\}} - M\binom{\{3\}}{\{3\}} + M\binom{\{2,3\}}{\{2,3\}}
\end{aligned}$$

Formal definitions:
Let $R \subseteq \{2,3\}$ and $C \subseteq \{2,3\}$ with $|R| = |C|$.

$$\Delta\binom{R}{C} = \det A_{\{1\}\cup R,\{1\}\cup C} \qquad M(S) = \prod_{(R,C)\in S} \Delta\binom{R}{C} \cdot \prod_{(R,C)\notin S} \Gamma\binom{R}{C}$$

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$$\begin{aligned}
a_{11} b_6 &= a_{11}\,(a_{11}a_{22} - a_{12}a_{21})\,(a_{11}a_{23} - a_{13}a_{21})\,(a_{11}a_{32} - a_{12}a_{31})\,(a_{11}a_{33} - a_{13}a_{31}) \\
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&= \Delta\binom{\{\}}{\{\}}\,\Delta\binom{\{2\}}{\{2\}}\,\Delta\binom{\{2\}}{\{3\}}\,\Delta\binom{\{3\}}{\{2\}}\,\Delta\binom{\{3\}}{\{3\}}\,\Delta\binom{\{2,3\}}{\{2,3\}} \\
&= M\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix}
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a_{11} b_0 &= a_{11}^6 a_{22}a_{23}a_{32}a_{33}\,(a_{22}a_{33} - a_{23}a_{32}) \\
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&= M\Big(\Big)
\end{aligned}$$

$$\begin{aligned}
a_{11} b_1 &= a_{11}^5\,(-6a_{11}a_{22}^2a_{23}a_{32}a_{33}^2 + 6a_{11}a_{22}a_{23}^2a_{32}^2a_{33} - a_{12}a_{21}a_{23}^2a_{32}^2a_{33} \\
&\quad + a_{12}a_{22}^2a_{23}a_{31}a_{33}^2 + a_{13}a_{21}a_{22}^2a_{32}a_{33}^2 - a_{13}a_{22}a_{23}^2a_{31}a_{32}^2) \\
&= -3M\binom{\{\}}{\{\}} - M\binom{\{2\}}{\{2\}} - M\binom{\{2\}}{\{3\}} - M\binom{\{3\}}{\{2\}} - M\binom{\{3\}}{\{3\}} + M\binom{\{2,3\}}{\{2,3\}} \\
&= -3\Sigma\binom{\{\}}{\{\}} - \Sigma\binom{\{2\}}{\{2\}} + \Sigma\binom{\{2,3\}}{\{2,3\}}
\end{aligned}$$

Theorem (Rowland–Wu 2025)

Let A be a 3×3 matrix with positive entries.
The top left entry x of $\mathrm{Sink}(A)$ *satisfies* $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, *where*

$$a_{11} b_6 = \Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}\ \{3\}\ \{2,3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}\ \{3\}\ \{2,3\}}$$

$$a_{11} b_5 = -3\Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}\ \{3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}\ \{3\}} - \Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}\ \{2,3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}\ \{2,3\}} + \Sigma\binom{\{2\}\ \{2\}\ \{3\}\ \{3\}\ \{2,3\}}{\{2\}\ \{3\}\ \{2\}\ \{3\}\ \{2,3\}}$$

$$a_{11} b_4 = 4\Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}} + \Sigma\binom{\{\}\ \{2\}\ \{3\}\ \{2,3\}}{\{\}\ \{2\}\ \{3\}\ \{2,3\}} - 3\Sigma\binom{\{2\}\ \{2\}\ \{3\}\ \{3\}}{\{2\}\ \{3\}\ \{2\}\ \{3\}}$$

$$a_{11} b_3 = -4\Sigma\binom{\{\}\ \{2\}\ \{2\}}{\{\}\ \{2\}\ \{3\}} - 5\Sigma\binom{\{\}\ \{2\}\ \{3\}}{\{\}\ \{2\}\ \{3\}} + \Sigma\binom{\{\}\ \{2\}\ \{2,3\}}{\{\}\ \{2\}\ \{2,3\}} + \Sigma\binom{\{2\}\ \{2\}\ \{3\}}{\{2\}\ \{3\}\ \{2\}} - \Sigma\binom{\{2\}\ \{3\}\ \{2,3\}}{\{2\}\ \{3\}\ \{2,3\}}$$

$$a_{11} b_2 = 4\Sigma\binom{\{\}\ \{2\}}{\{\}\ \{2\}} - 3\Sigma\binom{\{\}\ \{2,3\}}{\{\}\ \{2,3\}} + \Sigma\binom{\{2\}\ \{3\}}{\{2\}\ \{3\}}$$

$$a_{11} b_1 = -3\Sigma\binom{\{\}}{\{\}} - \Sigma\binom{\{2\}}{\{2\}} + \Sigma\binom{\{2,3\}}{\{2,3\}}$$

$$a_{11} b_0 = \Sigma\big(\big)\,.$$

The top left entry x of $\mathrm{Sink}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$ satisfies $(ad - bc)x^2 - 2adx + ad = 0$.

$$\Sigma\binom{\{\}\ \{2\}}{\{\}\ \{2\}} x^2 - 2\Sigma\binom{\{\}}{\{\}} x + \Sigma\big(\big) = 0$$

Degree: number of minors not involving the first row or first column.

For $n \times n$ matrices: $\displaystyle\sum_{j=0}^{n-1} \binom{n-1}{j}^2 = \binom{2n-2}{n-1}$ $\quad 1, 2, 6, 20, 70, 252, \ldots$

What are the integer coefficients?

Gröbner basis computations are infeasible for $n \geq 4$.
Instead, use PSLQ and interpolate from examples.

To get more data:

Definition

Let A be an $m \times n$ matrix with positive entries.
The *Sinkhorn limit* of A is obtained by iteratively scaling so that each row sum is 1 and each column sum is $\frac{m}{n}$.

Existence (in a more general form): Sinkhorn 1967.

1.5 CPU years scaling matrices and recognizing 102K algebraic numbers let us solve for 63K coefficients (and 56K parameterized by free variables).

Conjecture (Rowland–Wu 2025)

Let $m \geq 1$ and $n \geq 1$. For every $m \times n$ matrix A with positive entries, the top left entry x of $\operatorname{Sink}(A)$ satisfies

$$\sum_{S \subseteq D(m,n)} \det\left(\tfrac{1}{m} \operatorname{adj}_S(m,n)\right) M(S) x^{|S|} = 0.$$

In particular, x is algebraic over the field generated by the entries of A, with degree at most $\binom{m+n-2}{m-1}$.

Set of minor specifications not involving the first row or column:
$D(m,n) = \{(R, C) : R \subseteq \{2, \dots, m\} \text{ and } C \subseteq \{2, \dots, n\} \text{ and } |R| = |C|\}$

$\operatorname{adj}_S(m,n)$ is an adjacency-like matrix with entries that are linear in m, n.

Example

Let $S = \begin{smallmatrix} \{\} \\ \{\} \end{smallmatrix}$.

Table of coefficients of $M\begin{pmatrix} \{\} \\ \{\} \end{pmatrix}$:

	$n = 1$	2	3	4
$m = 1$				
2				
3				
4				

2×2: $\quad \Sigma\begin{pmatrix} \{\} & \{2\} \\ \{\} & \{2\} \end{pmatrix} x^2 - 2\Sigma\begin{pmatrix} \{\} \\ \{\} \end{pmatrix} x + \Sigma\begin{pmatrix} \end{pmatrix} = 0$

3×3: $\quad a_{11} b_1 = -3\Sigma\begin{pmatrix} \{\} \\ \{\} \end{pmatrix} - \Sigma\begin{pmatrix} \{2\} \\ \{2\} \end{pmatrix} + \Sigma\begin{pmatrix} \{2,3\} \\ \{2,3\} \end{pmatrix}$

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3			-3	
4				-4

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Example

Let $S = \begin{smallmatrix}\{\}\\\{\}\end{smallmatrix}$.

Table of coefficients of $M\begin{pmatrix}\{\}\\\{\}\end{pmatrix}$:

	$n=1$	2	3	4
$m=1$	-1	-2	-3	-4
2	-1	-2	-3	-4
3	-1	-2	-3	-4
4	-1	-2	-3	-4

2×2: $\quad \Sigma\begin{pmatrix}\{\} & \{2\}\\\{\} & \{2\}\end{pmatrix} x^2 - 2\Sigma\begin{pmatrix}\{\}\\\{\}\end{pmatrix} x + \Sigma\begin{pmatrix}\end{pmatrix} = 0$

3×3: $\quad a_{11} b_1 = -3\Sigma\begin{pmatrix}\{\}\\\{\}\end{pmatrix} - \Sigma\begin{pmatrix}\{2\}\\\{2\}\end{pmatrix} + \Sigma\begin{pmatrix}\{2{,}3\}\\\{2{,}3\}\end{pmatrix}$

Example

Let $S = \begin{smallmatrix}\{\} \\ \{\}\end{smallmatrix}$.

Table of coefficients of $M\left(\begin{smallmatrix}\{\} \\ \{\}\end{smallmatrix}\right)$:

	$n=1$	2	3	4
$m=1$	-1	-2	-3	-4
2	-1	-2	-3	-4
3	-1	-2	-3	-4
4	-1	-2	-3	-4

Seems to be $-n$

2×2: $\quad \Sigma\left(\begin{smallmatrix}\{\} & \{2\} \\ \{\} & \{2\}\end{smallmatrix}\right) x^2 - 2\Sigma\left(\begin{smallmatrix}\{\} \\ \{\}\end{smallmatrix}\right) x + \Sigma\left(\right) = 0$

3×3: $\quad a_{11} b_1 = -3\Sigma\left(\begin{smallmatrix}\{\} \\ \{\}\end{smallmatrix}\right) - \Sigma\left(\begin{smallmatrix}\{2\} \\ \{2\}\end{smallmatrix}\right) + \Sigma\left(\begin{smallmatrix}\{2,3\} \\ \{2,3\}\end{smallmatrix}\right)$

Example

Let $S = \begin{smallmatrix} \{\} \\ \{\} \end{smallmatrix}$.

Table of coefficients of $M\begin{pmatrix} \{\} \\ \{\} \end{pmatrix}$:

	$n = 1$	2	3	4
$m = 1$	-1	-2	-3	-4
2	-1	-2	-3	-4
3	-1	-2	-3	-4
4	-1	-2	-3	-4

Seems to be $-n = \det \left[-n\right]$

2×2: $\quad \Sigma\begin{pmatrix} \{\} & \{2\} \\ \{\} & \{2\} \end{pmatrix} x^2 - 2\Sigma\begin{pmatrix} \{\} \\ \{\} \end{pmatrix} x + \Sigma\begin{pmatrix} \end{pmatrix} = 0$

3×3: $\quad a_{11} b_1 = -3\Sigma\begin{pmatrix} \{\} \\ \{\} \end{pmatrix} - \Sigma\begin{pmatrix} \{2\} \\ \{2\} \end{pmatrix} + \Sigma\begin{pmatrix} \{2,3\} \\ \{2,3\} \end{pmatrix}$

Example

Let $S = \begin{smallmatrix} \{\} \\ \{\} \end{smallmatrix}$.

Table of coefficients of $M\binom{\{\}}{\{\}}$:

	$n=1$	2	3	4
$m=1$	-1	-2	-3	-4
2	-1	-2	-3	-4
3	-1	-2	-3	-4
4	-1	-2	-3	-4

Seems to be $-n = \det\left[-n\right] = \det\left(\frac{1}{m}\left[-mn\right]\right)$.

2×2: $\quad \Sigma\begin{pmatrix} \{\} & \{2\} \\ \{\} & \{2\} \end{pmatrix} x^2 - 2\Sigma\begin{pmatrix} \{\} \\ \{\} \end{pmatrix} x + \Sigma\left(\right) = 0$

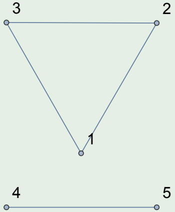
3×3: $\quad a_{11} b_1 = -3\Sigma\begin{pmatrix} \{\} \\ \{\} \end{pmatrix} - \Sigma\begin{pmatrix} \{2\} \\ \{2\} \end{pmatrix} + \Sigma\begin{pmatrix} \{2,3\} \\ \{2,3\} \end{pmatrix}$

Recall

The adjacency matrix of a k-vertex graph is the $k \times k$ matrix with entries

$$a_{ij} = \begin{cases} 1 & \text{if vertices } i, j \text{ are connected by an edge} \\ 0 & \text{if not.} \end{cases}$$

Example

$G =$ 

$$\operatorname{adj}(G) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

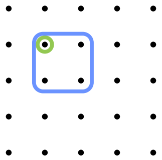
Connected components:

$$\det \operatorname{adj}(G_1 + G_2) = \det \operatorname{adj}(G_1) \cdot \det \operatorname{adj}(G_2)$$

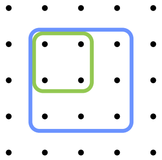
Underlying graph: Vertex set S. What are the edges/links?

Type-1 links: Sizes differ by 1, and one is a subset of the other.

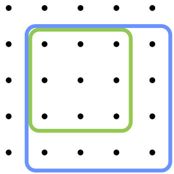
$$S = \begin{matrix} \{2\} & \{2,3\} \\ \{2\} & \{2,3\} \end{matrix}$$



$$S = \begin{matrix} \{2,3\} & \{2,3,4\} \\ \{2,3\} & \{2,3,4\} \end{matrix}$$

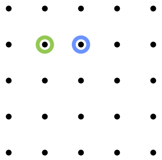


$$S = \begin{matrix} \{2,3,4\} & \{2,3,4,5\} \\ \{2,3,4\} & \{2,3,4,5\} \end{matrix}$$

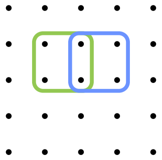


Type-2 links: Same sizes, and they differ in exactly 1 row or 1 column.

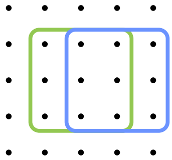
$$S = \begin{matrix} \{2\} & \{2\} \\ \{2\} & \{3\} \end{matrix}$$



$$S = \begin{matrix} \{2,3\} & \{2,3\} \\ \{2,3\} & \{3,4\} \end{matrix}$$



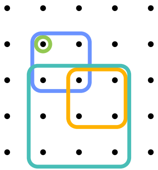
$$S = \begin{matrix} \{2,3,4\} & \{2,3,4\} \\ \{2,3,4\} & \{3,4,5\} \end{matrix}$$



Connected components are built from these.

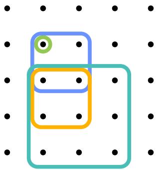
Two components:

$$S = \begin{matrix} \{2\} & \{2,3\} & \{3,4\} & \{3,4,5\} \\ \{2\} & \{2,3\} & \{3,4\} & \{2,3,4\} \end{matrix}$$



One component:

$$S = \begin{matrix} \{2\} & \{2,3\} & \{3,4\} & \{3,4,5\} \\ \{2\} & \{2,3\} & \{2,3\} & \{2,3,4\} \end{matrix}$$



$\mathrm{adj}_S(m, n)$ should be determined by linked pairs and singletons.

Definition

Write $S = \begin{smallmatrix} R_1 & R_2 & \cdots & R_k \\ C_1 & C_2 & \cdots & C_k \end{smallmatrix}$. The $k \times k$ matrix $\mathrm{adj}_S(m, n)$ is as follows.

(i, i) diagonal entry:

$$|R_i|(m + n) - mn$$

Let $\tau_{ij} = (R_i \setminus R_j,\ R_j \setminus R_i,\ C_i \setminus C_j,\ C_j \setminus C_i)$.

(i, j) off-diagonal entry:

$$\begin{cases} \pm m & \text{if } \tau_{ij} = (\{\}, \{s\}, \{\}, \{t\}) \text{ for some } s, t \\ \pm n & \text{if } \tau_{ij} = (\{s\}, \{\}, \{t\}, \{\}) \text{ for some } s, t \\ \pm m & \text{if } \tau_{ij} = (\{\}, \{\}, \{s\}, \{t\}) \text{ for some } s, t \\ \pm n & \text{if } \tau_{ij} = (\{s\}, \{t\}, \{\}, \{\}) \text{ for some } s, t \\ 0 & \text{otherwise} \end{cases}$$

Definition

Write $S = \begin{smallmatrix} R_1 & R_2 & \cdots & R_k \\ C_1 & C_2 & \cdots & C_k \end{smallmatrix}$. The $k \times k$ matrix $\mathrm{adj}_S(m, n)$ is as follows.

(i, i) diagonal entry:

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(i, j) off-diagonal entry:

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What are the signs?

They aren't determined by the underlying graph.

5-cycles that require different signs:

Example

$$S = \begin{matrix} \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} \\ \{2,3,4\} & \{2,3,5\} & \{2,5,7\} & \{2,6,7\} & \{2,4,6\} \end{matrix}$$

$$\operatorname{adj}_S(m,n) = \begin{bmatrix} 3m+3n-mn & m & 0 & 0 & -m \\ m & 3m+3n-mn & -m & 0 & 0 \\ 0 & -m & 3m+3n-mn & m & 0 \\ 0 & 0 & m & 3m+3n-mn & -m \\ -m & 0 & 0 & -m & 3m+3n-mn \end{bmatrix}$$

Example

$$T = \begin{matrix} \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} \\ \{2,3,4\} & \{2,3,5\} & \{3,5,6\} & \{4,5,6\} & \{2,4,6\} \end{matrix}$$

$$\operatorname{adj}_T(m,n) = \begin{bmatrix} 3m+3n-mn & m & 0 & 0 & -m \\ m & 3m+3n-mn & m & 0 & 0 \\ 0 & m & 3m+3n-mn & m & 0 \\ 0 & 0 & m & 3m+3n-mn & -m \\ -m & 0 & 0 & -m & 3m+3n-mn \end{bmatrix}$$

5-cycles that require different signs:

Example

$S =$ {2,3,4} {2,3,4} {2,3,4} {2,3,4} {2,3,4}
{2,3,4} {2,3,5} {2,5,7} {2,6,7} {2,4,6}

$$\operatorname{adj}_S(m,n) = \begin{bmatrix} 3m+3n-mn & m & 0 & 0 & -m \\ m & 3m+3n-mn & -m & 0 & 0 \\ 0 & -m & 3m+3n-mn & m & 0 \\ 0 & 0 & m & 3m+3n-mn & -m \\ -m & 0 & 0 & -m & 3m+3n-mn \end{bmatrix}$$

Permutations:

$$\{2,3,4\} \mapsto \{2,3,5\} \mapsto \{2,5,7\} \mapsto \{2,6,7\} \mapsto \{2,4,6\} \mapsto \{2,3,4\}$$

Example

$T =$ {2,3,4} {2,3,4} {2,3,4} {2,3,4} {2,3,4}
{2,3,4} {2,3,5} {3,5,6} {4,5,6} {2,4,6}

$$\operatorname{adj}_T(m,n) = \begin{bmatrix} 3m+3n-mn & m & 0 & 0 & -m \\ m & 3m+3n-mn & m & 0 & 0 \\ 0 & m & 3m+3n-mn & m & 0 \\ 0 & 0 & m & 3m+3n-mn & -m \\ -m & 0 & 0 & -m & 3m+3n-mn \end{bmatrix}$$

5-cycles that require different signs:

Example

$S = \begin{matrix} \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} \\ \{2,3,4\} & \{2,3,5\} & \{2,5,7\} & \{2,6,7\} & \{2,4,6\} \end{matrix}$

$$\operatorname{adj}_S(m,n) = \begin{bmatrix} 3m+3n-mn & m & 0 & 0 & -m \\ m & 3m+3n-mn & -m & 0 & 0 \\ 0 & -m & 3m+3n-mn & m & 0 \\ 0 & 0 & m & 3m+3n-mn & -m \\ -m & 0 & 0 & -m & 3m+3n-mn \end{bmatrix}$$

Permutations:

$$\{2,3,4\} \overset{+}{\mapsto} \{2,3,5\} \overset{-}{\mapsto} \{2,5,7\} \overset{+}{\mapsto} \{2,6,7\} \overset{-}{\mapsto} \{2,4,6\} \overset{-}{\mapsto} \{2,3,4\}$$

Example

$T = \begin{matrix} \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} \\ \{2,3,4\} & \{2,3,5\} & \{3,5,6\} & \{4,5,6\} & \{2,4,6\} \end{matrix}$

$$\operatorname{adj}_T(m,n) = \begin{bmatrix} 3m+3n-mn & m & 0 & 0 & -m \\ m & 3m+3n-mn & m & 0 & 0 \\ 0 & m & 3m+3n-mn & m & 0 \\ 0 & 0 & m & 3m+3n-mn & -m \\ -m & 0 & 0 & -m & 3m+3n-mn \end{bmatrix}$$

5-cycles that require different signs:

Example

$$S = \begin{matrix} \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} \\ \{2,3,4\} & \{2,3,5\} & \{2,5,7\} & \{2,6,7\} & \{2,4,6\} \end{matrix}$$

$$\mathrm{adj}_S(m,n) = \begin{bmatrix} 3m+3n-mn & m & 0 & 0 & -m \\ m & 3m+3n-mn & -m & 0 & 0 \\ 0 & -m & 3m+3n-mn & m & 0 \\ 0 & 0 & m & 3m+3n-mn & -m \\ -m & 0 & 0 & -m & 3m+3n-mn \end{bmatrix}$$

Permutations:

$$\{2,3,4\} \overset{+}{\mapsto} \{2,3,5\} \overset{-}{\mapsto} \{2,5,7\} \overset{+}{\mapsto} \{2,6,7\} \overset{-}{\mapsto} \{2,4,6\} \overset{-}{\mapsto} \{2,3,4\}$$

Example

$$T = \begin{matrix} \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} \\ \{2,3,4\} & \{2,3,5\} & \{3,5,6\} & \{4,5,6\} & \{2,4,6\} \end{matrix}$$

$$\mathrm{adj}_T(m,n) = \begin{bmatrix} 3m+3n-mn & m & 0 & 0 & -m \\ m & 3m+3n-mn & m & 0 & 0 \\ 0 & m & 3m+3n-mn & m & 0 \\ 0 & 0 & m & 3m+3n-mn & -m \\ -m & 0 & 0 & -m & 3m+3n-mn \end{bmatrix}$$

Permutations:

$$\{2,3,4\} \mapsto \{2,3,5\} \mapsto \{3,5,6\} \mapsto \{4,5,6\} \mapsto \{2,4,6\} \mapsto \{2,3,4\}$$

5-cycles that require different signs:

Example

$S = \begin{matrix} \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} \\ \{2,3,4\} & \{2,3,5\} & \{2,5,7\} & \{2,6,7\} & \{2,4,6\} \end{matrix}$

$$\mathrm{adj}_S(m,n) = \begin{bmatrix} 3m+3n-mn & m & 0 & 0 & -m \\ m & 3m+3n-mn & -m & 0 & 0 \\ 0 & -m & 3m+3n-mn & m & 0 \\ 0 & 0 & m & 3m+3n-mn & -m \\ -m & 0 & 0 & -m & 3m+3n-mn \end{bmatrix}$$

Permutations:

$$\{2,3,4\} \overset{+}{\mapsto} \{2,3,5\} \overset{-}{\mapsto} \{2,5,7\} \overset{+}{\mapsto} \{2,6,7\} \overset{-}{\mapsto} \{2,4,6\} \overset{-}{\mapsto} \{2,3,4\}$$

Example

$T = \begin{matrix} \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} & \{2,3,4\} \\ \{2,3,4\} & \{2,3,5\} & \{3,5,6\} & \{4,5,6\} & \{2,4,6\} \end{matrix}$

$$\mathrm{adj}_T(m,n) = \begin{bmatrix} 3m+3n-mn & m & 0 & 0 & -m \\ m & 3m+3n-mn & m & 0 & 0 \\ 0 & m & 3m+3n-mn & m & 0 \\ 0 & 0 & m & 3m+3n-mn & -m \\ -m & 0 & 0 & -m & 3m+3n-mn \end{bmatrix}$$

Permutations:

$$\{2,3,4\} \overset{+}{\mapsto} \{2,3,5\} \overset{+}{\mapsto} \{3,5,6\} \overset{+}{\mapsto} \{4,5,6\} \overset{-}{\mapsto} \{2,4,6\} \overset{-}{\mapsto} \{2,3,4\}$$

Summary

Each entry of an $m \times n$ Sinkhorn limit is algebraic with degree $\leq \binom{m+n-2}{m-1}$ (the number of minor specifications not involving the first row or column).

The polynomial describing an entry is a linear combination of $M(S)x^{|S|}$ where S ranges over the subsets of minor specifications.

The coefficient of $M(S)x^{|S|}$ is the determinant of an adjacency-like matrix.

All of this is conjectural.

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