Combinatorial structure behind Sinkhorn limits

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Goal

Given a square matrix with positive entries, turn it into a "close" doubly stochastic matrix of the same size (row and column sums are 1).

2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

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Scale rows . . .

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2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows ...

$$\begin{bmatrix} .800000 & .200000 \\ .666667 & .333333 \end{bmatrix}$$

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2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns . . .

$$\begin{bmatrix} .545455 & .375000 \\ .454545 & .625000 \end{bmatrix}$$

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2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows . . .

$$\begin{bmatrix} .592593 & .407407 \\ .421053 & .578947 \end{bmatrix}$$

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2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .584615 & .413043 \\ .415385 & .586957 \end{bmatrix}$$

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2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on . . .

$$\begin{bmatrix} .585987 & .414013 \\ .414414 & .585586 \end{bmatrix}$$

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2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585752 & .414179 \\ .414248 & .585821 \end{bmatrix}$$

Goal

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2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585792 & .414208 \\ .414219 & .585781 \end{bmatrix}$$

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2×2 matrix:

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Scale rows, then columns, then rows, and so on . . .

$$\begin{bmatrix} .585785 & .414213 \\ .414215 & .585787 \end{bmatrix}$$

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$$\begin{bmatrix} .585786 & .414214 \\ .414214 & .585786 \end{bmatrix}$$

In the limit, we obtain the Sinkhorn limit of $\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$.

Sinkhorn 1964: The limit exists.

Goal

Given a square matrix with positive entries, turn it into a "close" doubly stochastic matrix of the same size (row and column sums are 1).

2×2 matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$$

Scale rows, then columns, then rows, and so on ...

$$\begin{bmatrix} .585786 & .414214 \\ .414214 & .585786 \end{bmatrix} \approx \begin{bmatrix} 2-\sqrt{2} & -1+\sqrt{2} \\ -1+\sqrt{2} & 2-\sqrt{2} \end{bmatrix}$$

In the limit, we obtain the Sinkhorn limit of $\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$.

Sinkhorn 1964: The limit exists.

Applications in computer science:

- preconditioning a linear system to improve numerical stability
- approximating the permanent of a matrix
- determining whether a graph has a perfect matching

Kalantari et al. and Wigderson et al. studied convergence, fast algorithms.

Applications in other areas:

- predicting telephone traffic (Kruithof 1937)
- transportation science (Deming–Stephan 1940)
- economics (Stone 1964)
- image processing (Herman–Lent 1976)
- operations research (Raghavan 1984)
- machine learning (Cuturi 2013)

Idel (2016) wrote a 100-page survey of Sinkhorn-related results.

Question

What are the exact entries of the Sinkhorn limit?

Notation:

$$\text{Sink}\left(\begin{bmatrix}4 & 1\\ 2 & 1\end{bmatrix}\right) = \begin{bmatrix}2-\sqrt{2} & -1+\sqrt{2}\\ -1+\sqrt{2} & 2-\sqrt{2}\end{bmatrix}$$

Theorem (Nathanson 2020)

For a 2×2 *matrix* $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ *with positive entries,*

$$\text{Sink}(A) = \frac{1}{\sqrt{ad}+\sqrt{bc}}\begin{bmatrix}\sqrt{ad} & \sqrt{bc}\\ \sqrt{bc} & \sqrt{ad}\end{bmatrix}.$$

The entries are algebraic.

The top left entry x satisfies $(ad - bc)x^2 - 2adx + ad = 0$.

For a symmetric 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ with positive entries:

Theorem (Ekhad–Zeilberger 2019)

The top left entry x of $\operatorname{Sink}(A)$ *satisfies* $c_4 x^4 + \cdots + c_1 x + c_0 = 0$*, where*

$$c_4 = -(a_{12}^2 - a_{11}a_{22})(a_{13}^2 - a_{11}a_{33})(-a_{11}a_{22}a_{33} + a_{11}a_{23}^2 + a_{12}^2 a_{33} - 2a_{12}a_{13}a_{23} + a_{13}^2 a_{22})$$

$$c_3 = (-4a_{11}^3 a_{22}^2 a_{33}^2 + 4a_{11}^3 a_{22} a_{23}^2 a_{33} + 4a_{11}^2 a_{12}^2 a_{22} a_{33}^2 - 3a_{11}^2 a_{12}^2 a_{23}^2 a_{33} - 2a_{11}^2 a_{12} a_{13} a_{22} a_{23} a_{33} + 4a_{11}^2 a_{13}^2 a_{22}^2 a_{33} - 3a_{11}^2 a_{13}^2 a_{22} a_{23}^2 - 2a_{11} a_{12}^2 a_{13}^2 a_{22} a_{33} + 2a_{11} a_{12}^2 a_{13}^2 a_{23}^2 - a_{12}^4 a_{13}^2 a_{33} + 2a_{12}^3 a_{13}^3 a_{23} - a_{12}^2 a_{13}^4 a_{22})$$

$$c_2 = a_{11}(6a_{11}^2 a_{22}^2 a_{33}^2 - 6a_{11}^2 a_{22} a_{23}^2 a_{33} - 2a_{11} a_{12}^2 a_{22} a_{33}^2 + 3a_{11} a_{12}^2 a_{23}^2 a_{33} - 2a_{11} a_{12} a_{13} a_{22} a_{23} a_{33} - 2a_{11} a_{13}^2 a_{22}^2 a_{33} + 3a_{11} a_{13}^2 a_{22} a_{23}^2 + 2a_{12}^3 a_{13} a_{23} a_{33} - 3a_{12}^2 a_{13}^2 a_{22} a_{33} - a_{12}^2 a_{13}^2 a_{23}^2 + 2a_{12} a_{13}^3 a_{22} a_{23})$$

$$c_1 = -a_{11}^2(4a_{11} a_{22}^2 a_{33}^2 - 4a_{11} a_{22} a_{23}^2 a_{33} + a_{12}^2 a_{23}^2 a_{33} - 2a_{12} a_{13} a_{22} a_{23} a_{33} + a_{13}^2 a_{22} a_{23}^2)$$

$$c_0 = a_{11}^3 a_{22} a_{33}(a_{22} a_{33} - a_{23}^2)$$

Computed with Gröbner bases.

For general 3×3 matrices, the Sinkhorn limit wasn't known!

$$\mathrm{Sink}\left(\begin{bmatrix} 2 & 4 & 3 \\ 1 & 8 & 8 \\ 7 & 3 & 1 \end{bmatrix}\right) \approx \begin{bmatrix} .250338 & .377025 & .372637 \\ .066831 & .402607 & .530562 \\ .682830 & .220368 & .096801 \end{bmatrix}$$

What are these numbers? Assume they're algebraic.

Compute an entry to high precision:
$x \approx .25033837405936848945454728685142925283386722173530167719 94$

Guess the degree. 6
Use the PSLQ integer relation algorithm to find a likely polynomial:

$$236379x^6 + 502124x^5 - 1610856x^4 + 19808x^3 + 661120x^2 - 94592x - 12288 = 0$$

Conjecture (Chen and Varghese 2019, Hofstra SSRP)

For 3×3 matrices A, the entries of $\mathrm{Sink}(A)$ have degree at most 6.

It suffices to describe the top left entry of $\mathrm{Sink}(A)$.

Fact

If we know one entry of $\mathrm{Sink}(A)$ *as a function of* A*, then we know them all.*

Reason: Iterative scaling isn't sensitive to row or column order.

For example, if we know

$$\mathrm{Sink}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{ad}}{\sqrt{ad}+\sqrt{bc}} & ? \\ ? & ? \end{bmatrix}$$

then its bottom left entry is the top left entry of

$$\mathrm{Sink}\left(\begin{bmatrix} c & d \\ a & b \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{cb}}{\sqrt{cb}+\sqrt{da}} & \\ & \end{bmatrix}.$$

For a 3×3 matrix, what is the top left entry of $\mathrm{Sink}(A)$? System of equations...

Row scaling — multiplication on the left.
Column scaling — multiplication on the right.

$$\mathrm{Sink}(A) = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} \qquad R = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad C = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}$$

9 equations from $\mathrm{Sink}(A) = RAC$:

$$\begin{aligned} s_{11} &= r_1 a_{11} c_1 & s_{12} &= r_1 a_{12} c_2 & s_{13} &= r_1 a_{13} c_3 \\ s_{21} &= r_2 a_{21} c_1 & s_{22} &= r_2 a_{22} c_2 & s_{23} &= r_2 a_{23} c_3 \\ s_{31} &= r_3 a_{31} c_1 & s_{32} &= r_3 a_{32} c_2 & s_{33} &= r_3 a_{33} c_3 \end{aligned}$$

6 equations from row and column sums:

$$\begin{aligned} s_{11} + s_{12} + s_{13} &= 1 & s_{11} + s_{21} + s_{31} &= 1 \\ s_{21} + s_{22} + s_{23} &= 1 & s_{12} + s_{22} + s_{32} &= 1 \\ s_{31} + s_{32} + s_{33} &= 1 & s_{13} + s_{23} + s_{33} &= 1 \end{aligned}$$

Want s_{11} in terms of a_{ij}.
15 equations; eliminate 14 variables $r_1, r_2, r_3, c_1, c_2, c_3, s_{12}, s_{13}, \ldots, s_{33}$.
Gröbner basis computation...

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$
\begin{aligned}
b_6 = {} & (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\
& \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\
b_5 = {} & -6a_{11}^5a_{22}^2a_{23}a_{32}a_{33}^2 + 6a_{11}^5a_{22}a_{23}^2a_{32}^2a_{33} + 8a_{11}^4a_{12}a_{21}a_{22}a_{23}a_{32}a_{33}^2 \\
& - 5a_{11}^4a_{12}a_{21}a_{23}^2a_{32}^2a_{33} + 5a_{11}^4a_{12}a_{22}^2a_{23}a_{31}a_{33}^2 - 8a_{11}^4a_{12}a_{22}a_{23}^2a_{31}a_{32}a_{33} \\
& + 5a_{11}^4a_{13}a_{21}a_{22}^2a_{32}a_{33}^2 - 8a_{11}^4a_{13}a_{21}a_{22}a_{23}a_{32}^2a_{33} + 8a_{11}^4a_{13}a_{22}^2a_{23}a_{31}a_{32}a_{33} \\
& - 5a_{11}^4a_{13}a_{22}a_{23}^2a_{31}a_{32}^2 - 2a_{11}^3a_{12}^2a_{21}^2a_{23}a_{32}a_{33}^2 - 6a_{11}^3a_{12}^2a_{21}a_{22}a_{23}a_{31}a_{33}^2 \\
& + 6a_{11}^3a_{12}^2a_{21}a_{23}^2a_{31}a_{32}a_{33} + 2a_{11}^3a_{12}^2a_{22}a_{23}^2a_{31}^2a_{33} - 6a_{11}^3a_{12}a_{13}a_{21}^2a_{22}a_{32}a_{33}^2 \\
& + 6a_{11}^3a_{12}a_{13}a_{21}^2a_{23}a_{32}^2a_{33} - 4a_{11}^3a_{12}a_{13}a_{21}a_{22}^2a_{31}a_{33}^2 + 4a_{11}^3a_{12}a_{13}a_{21}a_{23}^2a_{31}a_{32}^2 \\
& - 6a_{11}^3a_{12}a_{13}a_{22}^2a_{23}a_{31}^2a_{33} + 6a_{11}^3a_{12}a_{13}a_{22}a_{23}^2a_{31}^2a_{32} + 2a_{11}^3a_{13}^2a_{21}^2a_{22}a_{32}^2a_{33} \\
& - 6a_{11}^3a_{13}^2a_{21}a_{22}^2a_{31}a_{32}a_{33} + 6a_{11}^3a_{13}^2a_{21}a_{22}a_{23}a_{31}a_{32}^2 - 2a_{11}^3a_{13}^2a_{22}^2a_{23}a_{31}^2a_{32} \\
& + a_{11}^2a_{12}^3a_{21}^2a_{23}a_{31}a_{33}^2 - a_{11}^2a_{12}^3a_{21}a_{23}^2a_{31}^2a_{33} + a_{11}^2a_{12}^2a_{13}a_{21}^3a_{32}a_{33}^2 \\
& + 4a_{11}^2a_{12}^2a_{13}a_{21}^2a_{22}a_{31}a_{33}^2 - 4a_{11}^2a_{12}^2a_{13}a_{21}^2a_{23}a_{31}a_{32}a_{33} \\
& + 4a_{11}^2a_{12}^2a_{13}a_{21}a_{22}a_{23}a_{31}^2a_{33} - 4a_{11}^2a_{12}^2a_{13}a_{21}a_{23}^2a_{31}^2a_{32} - a_{11}^2a_{12}^2a_{13}a_{22}a_{23}^2a_{31}^3 \\
& - a_{11}^2a_{12}a_{13}^2a_{21}^3a_{32}^2a_{33} + 4a_{11}^2a_{12}a_{13}^2a_{21}^2a_{22}a_{31}a_{32}a_{33} - 4a_{11}^2a_{12}a_{13}^2a_{21}^2a_{23}a_{31}a_{32}^2 \\
& + 4a_{11}^2a_{12}a_{13}^2a_{21}a_{22}^2a_{31}^2a_{33} - 4a_{11}^2a_{12}a_{13}^2a_{21}a_{22}a_{23}a_{31}^2a_{32} + a_{11}^2a_{12}a_{13}^2a_{22}^2a_{23}a_{31}^3 \\
& - a_{11}^2a_{13}^3a_{21}^2a_{22}a_{31}a_{32}^2 + a_{11}^2a_{13}^3a_{21}a_{22}^2a_{31}^2a_{32} - 2a_{11}a_{12}^2a_{13}^2a_{21}^2a_{22}a_{31}^2a_{33} \\
& + 2a_{11}a_{12}^2a_{13}^2a_{21}^2a_{23}a_{31}^2a_{32} - a_{12}^3a_{13}^2a_{21}^3a_{31}^2a_{33} + a_{12}^3a_{13}^2a_{21}^2a_{23}a_{31}^3 \\
& + a_{12}^2a_{13}^3a_{21}^3a_{31}^2a_{32} - a_{12}^2a_{13}^3a_{21}^2a_{22}a_{31}^3
\end{aligned}
$$

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$
\begin{aligned}
b_4 = a_{11} \big(& 15a_{11}^4 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^4 a_{22} a_{23}^2 a_{32}^2 a_{33} - 12a_{11}^3 a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&+ 10a_{11}^3 a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 10a_{11}^3 a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 12a_{11}^3 a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&- 10a_{11}^3 a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 12a_{11}^3 a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 12a_{11}^3 a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&+ 10a_{11}^3 a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{11}^2 a_{12}^2 a_{21}^2 a_{23} a_{32} a_{33}^2 + 6a_{11}^2 a_{12}^2 a_{21} a_{22} a_{23} a_{31} a_{33}^2 \\
&- 6a_{11}^2 a_{12}^2 a_{21} a_{23}^2 a_{31} a_{32} a_{33} - a_{11}^2 a_{12}^2 a_{22} a_{23}^2 a_{31}^2 a_{33} + 6a_{11}^2 a_{12} a_{13} a_{21}^2 a_{22} a_{32} a_{33}^2 \\
&- 6a_{11}^2 a_{12} a_{13} a_{21}^2 a_{23} a_{32}^2 a_{33} + 6a_{11}^2 a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - 6a_{11}^2 a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 \\
&+ 6a_{11}^2 a_{12} a_{13} a_{22}^2 a_{23} a_{31}^2 a_{33} - 6a_{11}^2 a_{12} a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} - a_{11}^2 a_{13}^2 a_{21}^2 a_{22} a_{32}^2 a_{33} \\
&+ 6a_{11}^2 a_{13}^2 a_{21} a_{22}^2 a_{31} a_{32} a_{33} - 6a_{11}^2 a_{13}^2 a_{21} a_{22} a_{23} a_{31} a_{32}^2 + a_{11}^2 a_{13}^2 a_{22}^2 a_{23} a_{31}^2 a_{32} \\
&- 2a_{11} a_{12}^2 a_{13} a_{21}^2 a_{22} a_{31} a_{33}^2 + 2a_{11} a_{12}^2 a_{13} a_{21} a_{23}^2 a_{31}^2 a_{32} + 2a_{11} a_{12} a_{13}^2 a_{21}^2 a_{23} a_{31} a_{32}^2 \\
&- 2a_{11} a_{12} a_{13}^2 a_{21} a_{22}^2 a_{31}^2 a_{33} - 3a_{12}^2 a_{13}^2 a_{21}^2 a_{22} a_{31}^2 a_{33} + 3a_{12}^2 a_{13}^2 a_{21}^2 a_{23} a_{31}^2 a_{32} \big) \\
b_3 = 2a_{11}^2 \big(& -10a_{11}^3 a_{22}^2 a_{23} a_{32} a_{33}^2 + 10a_{11}^3 a_{22} a_{23}^2 a_{32}^2 a_{33} + 4a_{11}^2 a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&- 5a_{11}^2 a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} + 5a_{11}^2 a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 - 4a_{11}^2 a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&+ 5a_{11}^2 a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - 4a_{11}^2 a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} + 4a_{11}^2 a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&- 5a_{11}^2 a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 - a_{11} a_{12}^2 a_{21} a_{22} a_{23} a_{31} a_{33}^2 + a_{11} a_{12}^2 a_{21} a_{23}^2 a_{31} a_{32} a_{33} \\
&- a_{11} a_{12} a_{13} a_{21}^2 a_{22} a_{32} a_{33}^2 + a_{11} a_{12} a_{13} a_{21}^2 a_{23} a_{32}^2 a_{33} - 2a_{11} a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 \\
&+ 2a_{11} a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 - a_{11} a_{12} a_{13} a_{22}^2 a_{23} a_{31}^2 a_{33} + a_{11} a_{12} a_{13} a_{22} a_{23}^2 a_{31}^2 a_{32} \\
&- a_{11} a_{13}^2 a_{21} a_{22}^2 a_{31} a_{32} a_{33} + a_{11} a_{13}^2 a_{21} a_{22} a_{23} a_{31} a_{32}^2 + a_{12}^2 a_{13} a_{21}^2 a_{23} a_{31} a_{32} a_{33} \\
&- a_{12}^2 a_{13} a_{21} a_{22} a_{23} a_{31}^2 a_{33} - a_{12} a_{13}^2 a_{21}^2 a_{22} a_{31} a_{32} a_{33} + a_{12} a_{13}^2 a_{21} a_{22} a_{23} a_{31}^2 a_{32} \big)
\end{aligned}
$$

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$
\begin{aligned}
b_2 = a_{11}^3 \big(& 15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
& + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
& - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
& + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 \big) \\
b_1 = a_{11}^4 \big(& -6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
& + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 \big) \\
b_0 = a_{11}^5 & a_{22} a_{23} a_{32} a_{33} \left(a_{22} a_{33} - a_{23} a_{32} \right)
\end{aligned}
$$

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 &= a_{11}^3 \left(15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2\right. \\
&\quad + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&\quad - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&\quad \left. + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2\right) \\
b_1 &= a_{11}^4 \left(-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33}\right. \\
&\quad \left. + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2\right) \\
b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} \left(a_{22} a_{33} - a_{23} a_{32}\right)
\end{aligned}$$

Better formulation?

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$
\begin{aligned}
b_2 = {} & a_{11}^3 (15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
& + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
& - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
& + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
b_1 = {} & a_{11}^4 (-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
& + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
b_0 = {} & a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}
$$

Better formulation?

$$
\begin{aligned}
b_6 = {} & (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\
& \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31})
\end{aligned}
$$

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where. . .

$$\begin{aligned}
b_2 = {} & a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
& + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
& - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
& + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
b_1 = {} & a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
& + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
b_0 = {} & a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 = {} & (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
& \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
= {} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

b_6 is the product of 5 minors

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 = a_{11}^3 (&15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&+ 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&- 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&+ 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
b_1 = a_{11}^4 (&-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&+ a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
b_0 = a_{11}^5 & a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

b_6 is the product of 5 minors involving a_{11}.

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

b_6 is the product of 5 minors involving a_{11}.

b_0 is the product of 5 minors not involving a_{11} (and a_{11}^5).

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned} b_2 &= a_{11}^3 \left(15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \right. \\ &\quad + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\ &\quad - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\ &\quad \left. + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 \right) \\ b_1 &= a_{11}^4 \left(-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \right. \\ &\quad \left. + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 \right) \\ b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32}) \end{aligned}$$

Better formulation?

$$\begin{aligned} b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\ &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{aligned}$$

b_6 is the product of 5 minors involving a_{11} and 0 not.

b_0 is the product of 5 minors not involving a_{11} (and a_{11}^5).

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 = {} & a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
& + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
& - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
& + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
b_1 = {} & a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
& + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
b_0 = {} & a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 = {} & (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
& \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
= {} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

b_6 is the product of 5 minors involving a_{11} and 0 not.
b_0 is the product of 0 minors involving a_{11} and 5 not (and a_{11}^5).

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

Multiply each b_k by a_{11}.

- b_6 is the product of 5 minors involving a_{11} and 0 not.
- b_0 is the product of 0 minors involving a_{11} and 5 not (and a_{11}^5).

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 = a_{11}^3 (&15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&+ 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&- 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&+ 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
b_1 = a_{11}^4 (&-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&+ a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
b_0 = a_{11}^5 & a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

Multiply each b_k by a_{11}.

$a_{11} b_6$ is the product of 6 minors involving a_{11} and 0 not.

b_0 is the product of 0 minors involving a_{11} and 5 not (and a_{11}^5).

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 = {} & a_{11}^3 \left(15 a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15 a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2 a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \right. \\
& + 5 a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5 a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2 a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
& - 5 a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2 a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2 a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
& \left. + 5 a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2 \right) \\
b_1 = {} & a_{11}^4 \left(-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \right. \\
& \left. + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 \right) \\
b_0 = {} & a_{11}^5 a_{22} a_{23} a_{32} a_{33} \left(a_{22} a_{33} - a_{23} a_{32}\right)
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 = {} & (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
& \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
= {} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

Multiply each b_k by a_{11}.

$a_{11} b_6$ is the product of 6 minors involving a_{11} and 0 not.

$a_{11} b_0$ is the product of 0 minors involving a_{11} and 6 not (and a_{11}^6).

Theorem

The top left entry $x = s_{11}$ satisfies $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, where...

$$\begin{aligned}
b_2 &= a_{11}^3 (15a_{11}^2 a_{22}^2 a_{23} a_{32} a_{33}^2 - 15a_{11}^2 a_{22} a_{23}^2 a_{32}^2 a_{33} - 2a_{11} a_{12} a_{21} a_{22} a_{23} a_{32} a_{33}^2 \\
&\quad + 5a_{11} a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} - 5a_{11} a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + 2a_{11} a_{12} a_{22} a_{23}^2 a_{31} a_{32} a_{33} \\
&\quad - 5a_{11} a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 + 2a_{11} a_{13} a_{21} a_{22} a_{23} a_{32}^2 a_{33} - 2a_{11} a_{13} a_{22}^2 a_{23} a_{31} a_{32} a_{33} \\
&\quad + 5a_{11} a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2 + a_{12} a_{13} a_{21} a_{22}^2 a_{31} a_{33}^2 - a_{12} a_{13} a_{21} a_{23}^2 a_{31} a_{32}^2) \\
b_1 &= a_{11}^4 (-6a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\
&\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\
b_0 &= a_{11}^5 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32})
\end{aligned}$$

Better formulation?

$$\begin{aligned}
b_6 &= (a_{11} a_{22} - a_{12} a_{21})(a_{11} a_{23} - a_{13} a_{21})(a_{11} a_{32} - a_{12} a_{31})(a_{11} a_{33} - a_{13} a_{31}) \\
&\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\
&= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
\end{aligned}$$

Multiply each b_k by a_{11}.
$a_{11} b_6$ is the product of 6 minors involving a_{11} and 0 not.
$a_{11} b_0$ is the product of 0 minors involving a_{11} and 6 not (and a_{11}^6).
$a_{11} b_k$ involves products of k minors involving a_{11} and $6 - k$ not?

Let $R \subseteq \{2,3\}$ and $C \subseteq \{2,3\}$ with $|R| = |C|$. Define

$$\Delta\binom{R}{C} = \det A_{\{1\}\cup R, \{1\}\cup C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C} \qquad M(S) = \prod_{(R,C)\in S} \Delta\binom{R}{C} \cdot \prod_{(R,C)\notin S} \Gamma\binom{R}{C}$$

Coefficients:

$$a_{11} b_6 = a_{11} (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31})$$

Let $R \subseteq \{2,3\}$ and $C \subseteq \{2,3\}$ with $|R| = |C|$. Define

$$\Delta\binom{R}{C} = \det A_{\{1\}\cup R, \{1\}\cup C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C} \qquad M(S) = \prod_{(R,C)\in S} \Delta\binom{R}{C} \cdot \prod_{(R,C)\notin S} \Gamma\binom{R}{C}$$

Coefficients:

$$\begin{aligned} a_{11} b_6 &= a_{11} \left(a_{11} a_{22} - a_{12} a_{21}\right) \left(a_{11} a_{23} - a_{13} a_{21}\right) \left(a_{11} a_{32} - a_{12} a_{31}\right) \left(a_{11} a_{33} - a_{13} a_{31}\right) \\ &\quad \cdot \left(a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}\right) \\ &= \Delta\binom{\{\}}{\{\}} \Delta\binom{\{2\}}{\{2\}} \Delta\binom{\{2\}}{\{3\}} \Delta\binom{\{3\}}{\{2\}} \Delta\binom{\{3\}}{\{3\}} \Delta\binom{\{2,3\}}{\{2,3\}} \end{aligned}$$

Let $R \subseteq \{2,3\}$ and $C \subseteq \{2,3\}$ with $|R| = |C|$. Define

$$\Delta\binom{R}{C} = \det A_{\{1\}\cup R,\{1\}\cup C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C} \qquad M(S) = \prod_{(R,C)\in S} \Delta\binom{R}{C} \cdot \prod_{(R,C)\notin S} \Gamma\binom{R}{C}$$

Coefficients:

$$\begin{aligned} a_{11} b_6 &= a_{11}\,(a_{11}a_{22} - a_{12}a_{21})\,(a_{11}a_{23} - a_{13}a_{21})\,(a_{11}a_{32} - a_{12}a_{31})\,(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= \Delta\binom{\{\}}{\{\}}\,\Delta\binom{\{2\}}{\{2\}}\,\Delta\binom{\{2\}}{\{3\}}\,\Delta\binom{\{3\}}{\{2\}}\,\Delta\binom{\{3\}}{\{3\}}\,\Delta\binom{\{2,3\}}{\{2,3\}} \end{aligned}$$

$$a_{11} b_0 = a_{11}^6 a_{22} a_{23} a_{32} a_{33}\,(a_{22}a_{33} - a_{23}a_{32})$$

Let $R \subseteq \{2,3\}$ and $C \subseteq \{2,3\}$ with $|R| = |C|$. Define

$$\Delta\binom{R}{C} = \det A_{\{1\}\cup R,\{1\}\cup C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C} \qquad M(S) = \prod_{(R,C)\in S} \Delta\binom{R}{C} \cdot \prod_{(R,C)\notin S} \Gamma\binom{R}{C}$$

Coefficients:

$$\begin{aligned} a_{11} b_6 &= a_{11} (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= \Delta\binom{\{\}}{\{\}} \Delta\binom{\{2\}}{\{2\}} \Delta\binom{\{2\}}{\{3\}} \Delta\binom{\{3\}}{\{2\}} \Delta\binom{\{3\}}{\{3\}} \Delta\binom{\{2,3\}}{\{2,3\}} \end{aligned}$$

$$\begin{aligned} a_{11} b_0 &= a_{11}^6 a_{22}a_{23}a_{32}a_{33} (a_{22}a_{33} - a_{23}a_{32}) \\ &= \Gamma\binom{\{\}}{\{\}} \Gamma\binom{\{2\}}{\{2\}} \Gamma\binom{\{2\}}{\{3\}} \Gamma\binom{\{3\}}{\{2\}} \Gamma\binom{\{3\}}{\{3\}} \Gamma\binom{\{2,3\}}{\{2,3\}} \end{aligned}$$

Let $R \subseteq \{2,3\}$ and $C \subseteq \{2,3\}$ with $|R| = |C|$. Define

$$\Delta\binom{R}{C} = \det A_{\{1\}\cup R, \{1\}\cup C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C} \qquad M(S) = \prod_{(R,C)\in S} \Delta\binom{R}{C} \cdot \prod_{(R,C)\notin S} \Gamma\binom{R}{C}$$

Coefficients:

$$\begin{aligned} a_{11} b_6 &= a_{11} (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31})(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= \Delta\binom{\{\}}{\{\}} \Delta\binom{\{2\}}{\{2\}} \Delta\binom{\{2\}}{\{3\}} \Delta\binom{\{3\}}{\{2\}} \Delta\binom{\{3\}}{\{3\}} \Delta\binom{\{2,3\}}{\{2,3\}} \\ &= M\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} a_{11} b_0 &= a_{11}^6 a_{22}a_{23}a_{32}a_{33} (a_{22}a_{33} - a_{23}a_{32}) \\ &= \Gamma\binom{\{\}}{\{\}} \Gamma\binom{\{2\}}{\{2\}} \Gamma\binom{\{2\}}{\{3\}} \Gamma\binom{\{3\}}{\{2\}} \Gamma\binom{\{3\}}{\{3\}} \Gamma\binom{\{2,3\}}{\{2,3\}} \end{aligned}$$

Let $R \subseteq \{2,3\}$ and $C \subseteq \{2,3\}$ with $|R| = |C|$. Define

$$\Delta\binom{R}{C} = \det A_{\{1\}\cup R,\{1\}\cup C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C} \qquad\qquad M(S) = \prod_{(R,C)\in S} \Delta\binom{R}{C} \cdot \prod_{(R,C)\notin S} \Gamma\binom{R}{C}$$

Coefficients:

$$\begin{aligned} a_{11} b_6 &= a_{11}\,(a_{11}a_{22} - a_{12}a_{21})\,(a_{11}a_{23} - a_{13}a_{21})\,(a_{11}a_{32} - a_{12}a_{31})\,(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= \Delta\binom{\{\}}{\{\}}\,\Delta\binom{\{2\}}{\{2\}}\,\Delta\binom{\{2\}}{\{3\}}\,\Delta\binom{\{3\}}{\{2\}}\,\Delta\binom{\{3\}}{\{3\}}\,\Delta\binom{\{2,3\}}{\{2,3\}} \\ &= M\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} a_{11} b_0 &= a_{11}^6\, a_{22}a_{23}a_{32}a_{33}\,(a_{22}a_{33} - a_{23}a_{32}) \\ &= \Gamma\binom{\{\}}{\{\}}\,\Gamma\binom{\{2\}}{\{2\}}\,\Gamma\binom{\{2\}}{\{3\}}\,\Gamma\binom{\{3\}}{\{2\}}\,\Gamma\binom{\{3\}}{\{3\}}\,\Gamma\binom{\{2,3\}}{\{2,3\}} \\ &= M() \end{aligned}$$

Let $R \subseteq \{2,3\}$ and $C \subseteq \{2,3\}$ with $|R| = |C|$. Define

$$\Delta\binom{R}{C} = \det A_{\{1\}\cup R, \{1\}\cup C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C} \qquad M(S) = \prod_{(R,C)\in S} \Delta\binom{R}{C} \cdot \prod_{(R,C)\notin S} \Gamma\binom{R}{C}$$

Coefficients:

$$\begin{aligned} a_{11} b_6 &= a_{11}\,(a_{11}a_{22} - a_{12}a_{21})\,(a_{11}a_{23} - a_{13}a_{21})\,(a_{11}a_{32} - a_{12}a_{31})\,(a_{11}a_{33} - a_{13}a_{31}) \\ &\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\ &= \Delta\binom{\{\}}{\{\}}\,\Delta\binom{\{2\}}{\{2\}}\,\Delta\binom{\{2\}}{\{3\}}\,\Delta\binom{\{3\}}{\{2\}}\,\Delta\binom{\{3\}}{\{3\}}\,\Delta\binom{\{2,3\}}{\{2,3\}} \\ &= M\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} a_{11} b_0 &= a_{11}^6 a_{22}a_{23}a_{32}a_{33}\,(a_{22}a_{33} - a_{23}a_{32}) \\ &= \Gamma\binom{\{\}}{\{\}}\,\Gamma\binom{\{2\}}{\{2\}}\,\Gamma\binom{\{2\}}{\{3\}}\,\Gamma\binom{\{3\}}{\{2\}}\,\Gamma\binom{\{3\}}{\{3\}}\,\Gamma\binom{\{2,3\}}{\{2,3\}} \\ &= M\big(\big) \end{aligned}$$

$$\begin{aligned} a_{11} b_1 &= a_{11}^5\,(-6a_{11}a_{22}^2a_{23}a_{32}a_{33}^2 + 6a_{11}a_{22}a_{23}^2a_{32}^2a_{33} - a_{12}a_{21}a_{23}^2a_{32}^2a_{33} \\ &\quad + a_{12}a_{22}^2a_{23}a_{31}a_{33}^2 + a_{13}a_{21}a_{22}^2a_{32}a_{33}^2 - a_{13}a_{22}a_{23}^2a_{31}a_{32}^2) \end{aligned}$$

Let $R \subseteq \{2,3\}$ and $C \subseteq \{2,3\}$ with $|R| = |C|$. Define

$$\Delta\binom{R}{C} = \det A_{\{1\}\cup R,\{1\}\cup C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C} \qquad\qquad M(S) = \prod_{(R,C)\in S} \Delta\binom{R}{C} \cdot \prod_{(R,C)\notin S} \Gamma\binom{R}{C}$$

Coefficients:

$$\begin{aligned} a_{11} b_6 &= a_{11} (a_{11} a_{22} - a_{12} a_{21}) (a_{11} a_{23} - a_{13} a_{21}) (a_{11} a_{32} - a_{12} a_{31}) (a_{11} a_{33} - a_{13} a_{31}) \\ &\quad \cdot (a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}) \\ &= \Delta\binom{\{\}}{\{\}} \Delta\binom{\{2\}}{\{2\}} \Delta\binom{\{2\}}{\{3\}} \Delta\binom{\{3\}}{\{2\}} \Delta\binom{\{3\}}{\{3\}} \Delta\binom{\{2,3\}}{\{2,3\}} \\ &= M\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} a_{11} b_0 &= a_{11}^6 a_{22} a_{23} a_{32} a_{33} (a_{22} a_{33} - a_{23} a_{32}) \\ &= \Gamma\binom{\{\}}{\{\}} \Gamma\binom{\{2\}}{\{2\}} \Gamma\binom{\{2\}}{\{3\}} \Gamma\binom{\{3\}}{\{2\}} \Gamma\binom{\{3\}}{\{3\}} \Gamma\binom{\{2,3\}}{\{2,3\}} \\ &= M() \end{aligned}$$

$$\begin{aligned} a_{11} b_1 &= a_{11}^5 (-6 a_{11} a_{22}^2 a_{23} a_{32} a_{33}^2 + 6 a_{11} a_{22} a_{23}^2 a_{32}^2 a_{33} - a_{12} a_{21} a_{23}^2 a_{32}^2 a_{33} \\ &\quad + a_{12} a_{22}^2 a_{23} a_{31} a_{33}^2 + a_{13} a_{21} a_{22}^2 a_{32} a_{33}^2 - a_{13} a_{22} a_{23}^2 a_{31} a_{32}^2) \\ &= -3M\binom{\{\}}{\{\}} - M\binom{\{2\}}{\{2\}} - M\binom{\{2\}}{\{3\}} - M\binom{\{3\}}{\{2\}} - M\binom{\{3\}}{\{3\}} + M\binom{\{2,3\}}{\{2,3\}} \end{aligned}$$

Let $R \subseteq \{2,3\}$ and $C \subseteq \{2,3\}$ with $|R| = |C|$. Define

$$\Delta\binom{R}{C} = \det A_{\{1\}\cup R, \{1\}\cup C}$$

$$\Gamma\binom{R}{C} = a_{11} \det A_{R,C} \qquad M(S) = \prod_{(R,C)\in S} \Delta\binom{R}{C} \cdot \prod_{(R,C)\notin S} \Gamma\binom{R}{C}$$

Coefficients:

$$\begin{aligned}
a_{11} b_6 &= a_{11}\,(a_{11}a_{22} - a_{12}a_{21})\,(a_{11}a_{23} - a_{13}a_{21})\,(a_{11}a_{32} - a_{12}a_{31})\,(a_{11}a_{33} - a_{13}a_{31}) \\
&\quad \cdot (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}) \\
&= \Delta\binom{\{\}}{\{\}}\,\Delta\binom{\{2\}}{\{2\}}\,\Delta\binom{\{2\}}{\{3\}}\,\Delta\binom{\{3\}}{\{2\}}\,\Delta\binom{\{3\}}{\{3\}}\,\Delta\binom{\{2,3\}}{\{2,3\}} \\
&= M\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
a_{11} b_0 &= a_{11}^6 a_{22}a_{23}a_{32}a_{33}\,(a_{22}a_{33} - a_{23}a_{32}) \\
&= \Gamma\binom{\{\}}{\{\}}\,\Gamma\binom{\{2\}}{\{2\}}\,\Gamma\binom{\{2\}}{\{3\}}\,\Gamma\binom{\{3\}}{\{2\}}\,\Gamma\binom{\{3\}}{\{3\}}\,\Gamma\binom{\{2,3\}}{\{2,3\}} \\
&= M\big(\big)
\end{aligned}$$

$$\begin{aligned}
a_{11} b_1 &= a_{11}^5\,(-6a_{11}a_{22}^2a_{23}a_{32}a_{33}^2 + 6a_{11}a_{22}a_{23}^2a_{32}^2a_{33} - a_{12}a_{21}a_{23}^2a_{32}^2a_{33} \\
&\quad + a_{12}a_{22}^2a_{23}a_{31}a_{33}^2 + a_{13}a_{21}a_{22}^2a_{32}a_{33}^2 - a_{13}a_{22}a_{23}^2a_{31}a_{32}^2) \\
&= -3M\binom{\{\}}{\{\}} - M\binom{\{2\}}{\{2\}} - M\binom{\{2\}}{\{3\}} - M\binom{\{3\}}{\{2\}} - M\binom{\{3\}}{\{3\}} + M\binom{\{2,3\}}{\{2,3\}} \\
&= -3\Sigma\binom{\{\}}{\{\}} - \Sigma\binom{\{2\}}{\{2\}} + \Sigma\binom{\{2,3\}}{\{2,3\}}
\end{aligned}$$

$$\Sigma(S) = \sum_{T \equiv S} M(T)$$

Theorem (Rowland–Wu 2024)

Let A be a 3×3 matrix with positive entries.
The top left entry x of $\mathrm{Sink}(A)$ *satisfies* $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, *where*

$$
\begin{aligned}
a_{11} b_6 &= \Sigma\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix} \\
a_{11} b_5 &= -3\Sigma\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{3\} \end{pmatrix} - \Sigma\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2\} & \{2,3\} \end{pmatrix} + \Sigma\begin{pmatrix} \{2\} & \{2\} & \{3\} & \{3\} & \{2,3\} \\ \{2\} & \{3\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix} \\
a_{11} b_4 &= 4\Sigma\begin{pmatrix} \{\} & \{2\} & \{2\} & \{3\} \\ \{\} & \{2\} & \{3\} & \{2\} \end{pmatrix} + \Sigma\begin{pmatrix} \{\} & \{2\} & \{3\} & \{2,3\} \\ \{\} & \{2\} & \{3\} & \{2,3\} \end{pmatrix} - 3\Sigma\begin{pmatrix} \{2\} & \{2\} & \{3\} & \{3\} \\ \{2\} & \{3\} & \{2\} & \{3\} \end{pmatrix} \\
a_{11} b_3 &= -4\Sigma\begin{pmatrix} \{\} & \{2\} & \{2\} \\ \{\} & \{2\} & \{3\} \end{pmatrix} - 5\Sigma\begin{pmatrix} \{\} & \{2\} & \{3\} \\ \{\} & \{2\} & \{3\} \end{pmatrix} + \Sigma\begin{pmatrix} \{\} & \{2\} & \{2,3\} \\ \{\} & \{2\} & \{2,3\} \end{pmatrix} + \Sigma\begin{pmatrix} \{2\} & \{2\} & \{3\} \\ \{2\} & \{3\} & \{2\} \end{pmatrix} - \Sigma\begin{pmatrix} \{2\} & \{3\} & \{2,3\} \\ \{2\} & \{3\} & \{2,3\} \end{pmatrix} \\
a_{11} b_2 &= 4\Sigma\begin{pmatrix} \{\} & \{2\} \\ \{\} & \{2\} \end{pmatrix} - 3\Sigma\begin{pmatrix} \{\} & \{2,3\} \\ \{\} & \{2,3\} \end{pmatrix} + \Sigma\begin{pmatrix} \{2\} & \{3\} \\ \{2\} & \{3\} \end{pmatrix} \\
a_{11} b_1 &= -3\Sigma\begin{pmatrix} \{\} \\ \{\} \end{pmatrix} - \Sigma\begin{pmatrix} \{2\} \\ \{2\} \end{pmatrix} + \Sigma\begin{pmatrix} \{2,3\} \\ \{2,3\} \end{pmatrix} \\
a_{11} b_0 &= \Sigma() \,.
\end{aligned}
$$

Why these particular integer coefficients? (Why are they symmetric?)

Why degree 6? $1 + 4 + 1 = 6$ relevant minors.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

For $n \times n$ matrices: $\sum_{j=0}^{n-1} \binom{n-1}{j}^2 = \binom{2n-2}{n-1}$ relevant minors.

Conjecture

For $n \times n$ matrices A, the entries of $\mathrm{Sink}(A)$ have degree at most $\binom{2n-2}{n-1}$.

1×1: degree $\binom{0}{0} = 1$ $\quad x - 1 = 0$

2×2: degree $\binom{2}{1} = 2$ $\quad (ad - bc)x^2 - 2adx + ad = 0$

3×3: degree $\binom{4}{2} = 6$ $\quad$ (to reappear soon)

4×4: degree $\binom{6}{3} = 20$ $\quad$ Gröbner basis computation is infeasible.

5×5: degree $\binom{8}{4} = 70$

Central binomial coefficients.

What are the integer coefficients?

Interpolate from examples instead.

We have an explicit polynomial for 4×4 matrices.

Generalization:

Definition

Let A be an $m \times n$ matrix with positive entries.
The *Sinkhorn limit* of A is obtained by iteratively scaling so that each row sum is 1 and each column sum is $\frac{m}{n}$.

Existence (in a more general form): Sinkhorn 1967.

1.5 CPU years scaling matrices and recognizing 102K algebraic numbers let us solve for 63K coefficients (and 56K parameterized by free variables).

Theorem (Rowland–Wu 2024)

Let A be a 3×3 matrix with positive entries.
The top left entry x of $\mathrm{Sink}(A)$ *satisfies* $b_6 x^6 + \cdots + b_1 x + b_0 = 0$, *where*

$$a_{11} b_6 = \Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}\ \{3\}\ \{2,3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}\ \{3\}\ \{2,3\}}$$

$$a_{11} b_5 = -3\Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}\ \{3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}\ \{3\}} - \Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}\ \{2,3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}\ \{2,3\}} + \Sigma\binom{\{2\}\ \{2\}\ \{3\}\ \{3\}\ \{2,3\}}{\{2\}\ \{3\}\ \{2\}\ \{3\}\ \{2,3\}}$$

$$a_{11} b_4 = 4\Sigma\binom{\{\}\ \{2\}\ \{2\}\ \{3\}}{\{\}\ \{2\}\ \{3\}\ \{2\}} + \Sigma\binom{\{\}\ \{2\}\ \{3\}\ \{2,3\}}{\{\}\ \{2\}\ \{3\}\ \{2,3\}} - 3\Sigma\binom{\{2\}\ \{2\}\ \{3\}\ \{3\}}{\{2\}\ \{3\}\ \{2\}\ \{3\}}$$

$$a_{11} b_3 = -4\Sigma\binom{\{\}\ \{2\}\ \{2\}}{\{\}\ \{2\}\ \{3\}} - 5\Sigma\binom{\{\}\ \{2\}\ \{3\}}{\{\}\ \{2\}\ \{3\}} + \Sigma\binom{\{\}\ \{2\}\ \{2,3\}}{\{\}\ \{2\}\ \{2,3\}} + \Sigma\binom{\{2\}\ \{2\}\ \{3\}}{\{2\}\ \{3\}\ \{2\}} - \Sigma\binom{\{2\}\ \{3\}\ \{2,3\}}{\{2\}\ \{3\}\ \{2,3\}}$$

$$a_{11} b_2 = 4\Sigma\binom{\{\}\ \{2\}}{\{\}\ \{2\}} - 3\Sigma\binom{\{\}\ \{2,3\}}{\{\}\ \{2,3\}} + \Sigma\binom{\{2\}\ \{3\}}{\{2\}\ \{3\}}$$

$$a_{11} b_1 = -3\Sigma\binom{\{\}}{\{\}} - \Sigma\binom{\{2\}}{\{2\}} + \Sigma\binom{\{2,3\}}{\{2,3\}}$$

$$a_{11} b_0 = \Sigma\left(\right).$$

The coefficients seem to be determinants of adjacency-like matrices.

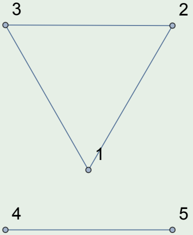
Recall

The adjacency matrix of an n-vertex graph is the $n \times n$ matrix with entries

$$a_{ij} = \begin{cases} 1 & \text{if vertices } i, j \text{ are connected by an edge} \\ 0 & \text{if not.} \end{cases}$$

Example

$G =$



$$\operatorname{adj}(G) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

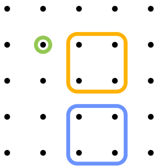
Connected components:

$$\det \operatorname{adj}(G_1 + G_2) = \det \operatorname{adj}(G_1) \cdot \det \operatorname{adj}(G_2)$$

$\text{adj}(S)$ is a $|S| \times |S|$ matrix.
Underlying graph: Vertex set S. What are the edges/links?

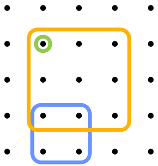
Unlinked minor specs (that nonetheless involve common rows/columns):

$$S = \begin{matrix} \{2\} & \{2,3\} & \{4,5\} \\ \{2\} & \{3,4\} & \{3,4\} \end{matrix}$$



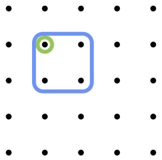
Unlinked minor specs (that nonetheless involve common entries):

$$S = \begin{matrix} \{2\} & \{4,5\} & \{2,3,4\} \\ \{2\} & \{2,3\} & \{2,3,4\} \end{matrix}$$

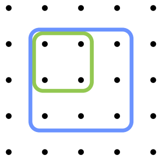


Type-1 links: Sizes differ by 1, and one is a subset of the other.

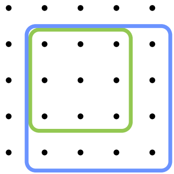
$$S = \begin{matrix} \{2\} & \{2,3\} \\ \{2\} & \{2,3\} \end{matrix}$$



$$S = \begin{matrix} \{2,3\} & \{2,3,4\} \\ \{2,3\} & \{2,3,4\} \end{matrix}$$

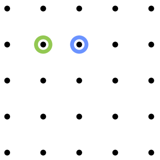


$$S = \begin{matrix} \{2,3,4\} & \{2,3,4,5\} \\ \{2,3,4\} & \{2,3,4,5\} \end{matrix}$$

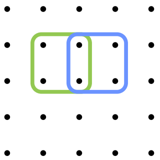


Type-2 links: Same sizes, and they differ in exactly 1 row or 1 column.

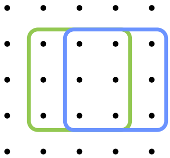
$$S = \begin{matrix} \{2\} & \{2\} \\ \{2\} & \{3\} \end{matrix}$$



$$S = \begin{matrix} \{2,3\} & \{2,3\} \\ \{2,3\} & \{3,4\} \end{matrix}$$



$$S = \begin{matrix} \{2,3,4\} & \{2,3,4\} \\ \{2,3,4\} & \{3,4,5\} \end{matrix}$$

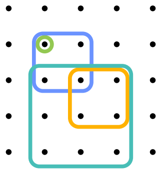


Type-1 links: Sizes differ by 1, and one is a subset of the other.
Type-2 links: Same sizes, and they differ in exactly 1 row or 1 column.

Connected components are built from these.

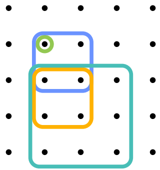
Two components:

$$S = \begin{matrix} \{2\} & \{2,3\} & \{3,4\} & \{3,4,5\} \\ \{2\} & \{2,3\} & \{3,4\} & \{2,3,4\} \end{matrix}$$



One component:

$$S = \begin{matrix} \{2\} & \{2,3\} & \{3,4\} & \{3,4,5\} \\ \{2\} & \{2,3\} & \{2,3\} & \{2,3,4\} \end{matrix}$$



To define adj(S), it suffices to define it for linked pairs and singletons.

$\mathrm{adj}_S(m,n)$ is a $|S| \times |S|$ matrix with entries that are linear in m, n.

Definition

Let $S = \substack{R_1 \\ C_1}\ \substack{R_2 \\ C_2}$. If $\substack{R_1 \\ C_1}$ and $\substack{R_2 \\ C_2}$ form a ...

- type-1 link with $|R_1| + 1 = |R_2|$,

$$\mathrm{adj}_S(m,n) := \begin{bmatrix} |R_1|(m+n) - mn & m \\ -n & |R_2|(m+n) - mn \end{bmatrix}.$$

- type-2 link with $R_1 = R_2$,

$$\mathrm{adj}_S(m,n) := \begin{bmatrix} |R_1|(m+n) - mn & -m \\ -m & |R_2|(m+n) - mn \end{bmatrix}.$$

- type-2 link with $C_1 = C_2$,

$$\mathrm{adj}_S(m,n) := \begin{bmatrix} |R_1|(m+n) - mn & -n \\ -n & |R_2|(m+n) - mn \end{bmatrix}.$$

If $\substack{R_1 \\ C_1}$ and $\substack{R_2 \\ C_2}$ are not linked,

$$\mathrm{adj}_S(m,n) := \begin{bmatrix} |R_1|(m+n) - mn & 0 \\ 0 & |R_2|(m+n) - mn \end{bmatrix}.$$

Example

For $S = \begin{smallmatrix} \{2\} & \{3\} & \{2,3\} \\ \{2\} & \{3\} & \{2,3\} \end{smallmatrix}$,

$$\mathrm{adj}_S(m,n) = \begin{bmatrix} m+n-mn & 0 & m \\ 0 & m+n-mn & m \\ -n & -n & 2m+2n-mn \end{bmatrix}.$$

This agrees with values we computed numerically.

Example

For $S = \begin{smallmatrix} \{2,3\} & \{2,3\} & \{2,3\} \\ \{2,3\} & \{2,4\} & \{2,5\} \end{smallmatrix}$,

$$\mathrm{adj}_S(m,n) = \begin{bmatrix} 2m+2n-mn & -m & -m \\ -m & 2m+2n-mn & -m \\ -m & -m & 2m+2n-mn \end{bmatrix}.$$

This does not agree with values we computed. A sign change is required:

$$\begin{bmatrix} 2m+2n-mn & m & m \\ m & 2m+2n-mn & m \\ m & m & 2m+2n-mn \end{bmatrix}$$

Summary

Each entry of an $m \times n$ Sinkhorn limit is algebraic with degree $\leq \binom{m+n-2}{m-1}$ (the number of minor specifications not involving the first row/column).

The polynomial describing an entry is a linear combination of $M(S)x^{|S|}$ where S ranges over the subsets of minor specifications.

The coefficient of $M(S)x^{|S|}$ is the determinant of an adjacency-like matrix. We don't know the signs of the off-diagonal entries.

All of this is conjectural.

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