Binomial coefficients and k-regular sequences

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Valuations of binomial coefficients

Pascal's triangle:

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

For this talk: p is a prime.

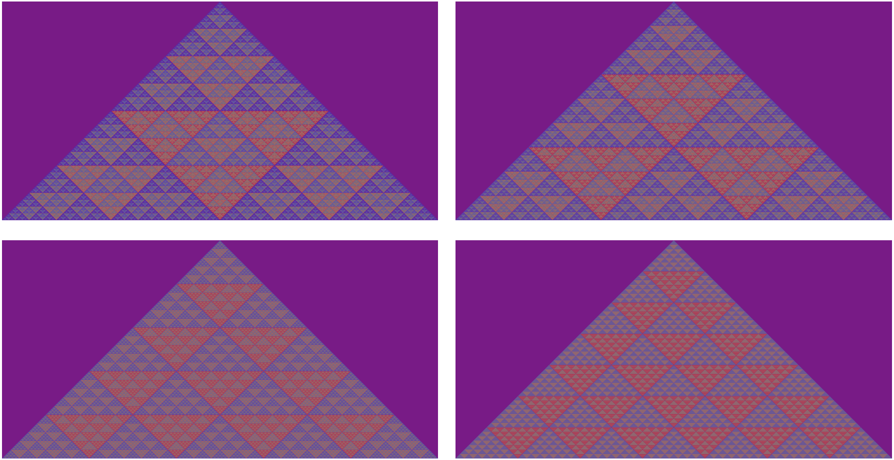
Let $\nu_p(n)$ denote the exponent of the highest power of p dividing n.
Example: $\nu_3(18) = 2$.

Theorem (Kummer 1852)

$\nu_p(\binom{n}{m}) =$ *number of carries involved in adding m to $n - m$ in base p.*

Valuations of binomial coefficients

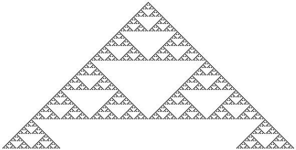
2-, 3-, 5-, and 7-adic valuations:



Odd binomial coefficients

Main theme: Arithmetic information about binomial coefficients reflects the base-p representations of integers.

$$\begin{array}{ccccccccc} &&&& 1 &&&& \\ &&& 1 && 1 &&& \\ && 1 && 2 && 1 && \\ & 1 && 3 && 3 && 1 & \\ 1 && 4 && 6 && 4 && 1 \end{array}$$



Glaisher (1899) counted odd binomial coefficients:

$$1, 2, 2, 4, 2, 4, 4, 8, 2, 4, 4, 8, 4, 8, 8, 16, \ldots \qquad \theta_{2,0}(n) = 2^{|n|_1}$$

Definition

$\theta_{p,\alpha}(n) := \left|\{m : 0 \leq m \leq n \text{ and } \nu_p(\binom{n}{m}) = \alpha\}\right|.$

$|n|_d :=$ number of occurrences of d in the base-p representation of n.

Derivation from Kummer's theorem

Glaisher's result $\theta_{2,0}(n) = 2^{|n|_1}$ follows from Kummer's theorem.

Theorem (Kummer)

$\nu_p(\binom{n}{m})$ = *number of carries involved in adding m to n − m in base p.*

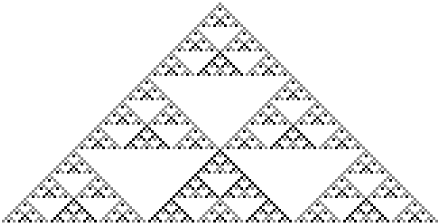
Example

$n = 25$. How many m satisfy $\nu_2(\binom{25}{m}) = 0$?

$$n = 25 = 11001_2$$
$$m = **00*_2$$

$\theta_{2,0}(25) = 2^{|25|_1} = 8.$

Binomial coefficients not divisible by p



Number of binomial coefficients with 3-adic valuation 0:

$$1, 2, 3, 2, 4, 6, 3, 6, 9, 2, 4, 6, 4, 8, 12, 6, \dots \qquad \theta_{3,0}(n) = 2^{|n|_1} 3^{|n|_2}$$

Theorem (Fine 1947)

Write $n = n_\ell \cdots n_1 n_0$ in base p. Then

$$\theta_{p,0}(n) = (n_0 + 1) \cdots (n_\ell + 1) = 1^{|n|_0} 2^{|n|_1} 3^{|n|_2} \cdots p^{|n|_{p-1}}.$$

Prime powers?

Carlitz found a recurrence involving $\theta_{p,\alpha}(n)$ and a secondary quantity $\psi_{p,\alpha}(n) := \left|\{m : 0 \leq m \leq n \text{ and } \nu_p((m+1)\binom{n}{m}) = \alpha\}\right|$.

Theorem (Carlitz 1967)

$$\theta_{p,\alpha}(pn+d) = (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1)$$

$$\psi_{p,\alpha}(pn+d) = \begin{cases} (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1) & \text{if } 0 \leq d \leq p-2 \\ p\psi_{p,\alpha-1}(n) & \text{if } d = p-1. \end{cases}$$

Is there a better formulation of this recurrence?

k-regular sequences

Constant-recursive sequences

Fibonacci recurrence: $F(n+2) = F(n+1) + F(n)$

Matrix form: $$\begin{bmatrix} F(n+1) \\ F(n+2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F(n) \\ F(n+1) \end{bmatrix}$$

Matrix product: $$F(n) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Characterizations of constant-recursive sequences over $\mathbb{Q}$:

- $s(n)$ is determined by a linear recurrence in $s(n+i)$ (along with finitely many initial conditions)
- $\langle \{ s(n+i)_{n \geq 0} : i \geq 0 \} \rangle$ is finite-dimensional
- $s(n) = u\, M^n\, v$ for some matrix M and vectors u, v
- generating function $\sum_{n \geq 0} s(n) x^n$ is rational

k-regularity

Definition

Let $k \geq 2$.
A sequence $s(n)_{n \geq 0}$ is k-regular if the vector space generated by

$$\{s(k^e n + i)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq i \leq k^e - 1\}$$

is finite-dimensional.

Characterizations of k-regularity (Allouche & Shallit 1992):

- $s(n)$ is determined by finitely many linear recurrences in $s(k^e n + i)$ (along with finitely many initial conditions)
- $s(n) = u\, M(n_0)\, M(n_1)\, \cdots\, M(n_\ell)\, v$ for some $M(d)$ and vectors u, v
- generating function in k non-commuting variables is rational

Examples of k-regular sequences

- $\nu_p(n)$

$$\nu_2(n)_{n\geq 1}: \quad 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, \ldots$$

- $\nu_p(F(n))$
- k-automatic sequences
  (e.g., the Thue–Morse sequence $0, 1, 1, 0, 1, 0, 0, 1, \ldots$)
- polynomial and quasi-polynomial sequences
- sums and products of k-regular sequences

A k-regular sequence reflects the base-k representation of n, so many nested sequences are k-regular.

How to guess a recurrence?

Guessing a constant-recursive sequence

$\langle \{ s(n+i)_{n \geq 0} : i \geq 0 \} \rangle$ is finite-dimensional.

$s(n) = 2^n + n$:

$s(n)$:	$1, 3, 6, 11, 20, 37, \ldots$	basis element!
$s(n+1)$:	$3, 6, 11, 20, 37, 70, \ldots$	basis element!
$s(n+2)$:	$6, 11, 20, 37, 70, 135, \ldots$	basis element!
$s(n+3)$:	$11, 20, 37, 70, 135, 264, \ldots$	$= 2s(n) - 5s(n+1) + 4s(n+2)$

Matrix form:

$$\begin{bmatrix} s(n+1) \\ s(n+2) \\ s(n+3) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} s(n) \\ s(n+1) \\ s(n+2) \end{bmatrix}$$

Guessing a 2-regular sequence

$s(n) = \theta_{2,1}(n) =$ number of binomial coefficients $\binom{n}{m}$ with $\nu_2(\binom{n}{m}) = 1$:

$s(n)$:	$0, 0, 1, 0, 1, 2, 2, 0, \ldots$	basis element!
$s(2n+0)$:	$0, 1, 1, 2, 1, 4, 2, 4, \ldots$	basis element!
$s(2n+1)$:	$0, 0, 2, 0, 2, 4, 4, 0, \ldots$	$= 2s(n)$
$s(4n+0)$:	$0, 1, 1, 2, 1, 4, 2, 4, \ldots$	$= s(2n)$
$s(4n+2)$:	$1, 2, 4, 4, 4, 8, 8, 8, \ldots$	basis element!
$s(8n+2)$:	$1, 4, 4, 8, 4, 12, 8, 16, \ldots$	$= -2s(n) + 2s(2n) + s(4n+2)$
$s(8n+6)$:	$2, 4, 8, 8, 8, 16, 16, 16, \ldots$	$= 2s(4n+2)$

Matrix form:

$$\begin{bmatrix} s(2n) \\ s(4n) \\ s(8n+2) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix} = M(0) \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix}$$

$$\begin{bmatrix} s(2n+1) \\ s(4n+2) \\ s(8n+6) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix} = M(1) \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix}$$

An implementation in *Mathematica*

IntegerSequences is available from

https://people.hofstra.edu/Eric_Rowland/packages.html

```
In[1]:= Import["https://people.hofstra.edu/Eric_Rowland/packages/IntegerSequences.m"]

In[2]:= Table[Count[Table[IntegerExponent[Binomial[n, m], 2], {m, 0, n}], 1], {n, 0, 31}]

Out[2]= {0, 0, 1, 0, 1, 2, 2, 0, 1, 2, 4, 4, 2, 4, 4, 0, 1, 2, 4, 4, 4, 8, 8, 8, 2, 4, 8, 8, 4, 8, 8, 0}

In[3]:= FindRegularSequenceFunction[%, 2] // RegularSequenceMatrixForm
```

Out[3]= $\text{RegularSequence}\left[\{1, 0, 0\}, \left\{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}\right\}, \{0, 0, 1\}\right]$

Sequences of polynomials

Fibonacci numbers

Combinatorial interpretation: $F(n) = \#$ compositions of $n-1$ using $1, 2$.
$n = 5$: $1+1+1+1$ $1+1+2$ $1+2+1$ $2+1+1$ $2+2$

Refinement:

$$F(n, x) := \sum_{\substack{\text{compositions } \lambda \text{ of} \\ n-1 \text{ using } 1,2}} x^{|\lambda|_1}$$

The coefficient of x^α is the number of compositions with α 1s.

n	$F(n,x)$
0	0
1	1
2	x
3	$x^2 + 1$

n	$F(n,x)$
4	$x^3 + 2x$
5	$1x^4 + 3x^2 + 1$
6	$x^5 + 4x^3 + 3x$
7	$x^6 + 5x^4 + 6x^2 + 1$

In particular, $F(n, 1) = F(n)$.

Fibonacci polynomials

Recurrence:

$$F(n+2,x) = x\,F(n+1,x) + F(n,x)$$

Matrix form:

$$\begin{bmatrix} F(n+1,x) \\ F(n+2,x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & x \end{bmatrix} \begin{bmatrix} F(n,x) \\ F(n+1,x) \end{bmatrix}$$

Matrix product:

$$F(n,x) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & x \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Generating function

Define

$$\color{red}{T_p(n,x)} := \sum_{m=0}^{n} x^{\nu_p(\binom{n}{m})} = \sum_{\alpha \geq 0} \theta_{p,\alpha}(n) x^{\alpha}.$$

$\theta_{p,\alpha}(n)$ is the number of binomial coefficients with p-adic valuation α.

$p = 2$:

n	$T_2(n,x)$
0	1
1	2
2	$x+2$
3	4
4	$2x^2+x+2$
5	$2x+4$
6	x^2+2x+4
7	8

n	$T_2(n,x)$
8	$4x^3+2x^2+x+2$
9	$4x^2+2x+4$
10	$2x^3+x^2+4x+4$
11	$4x+8$
12	$2x^3+5x^2+2x+4$
13	$2x^2+4x+8$
14	x^3+2x^2+4x+8
15	16

In particular, $T_p(n,1) = n+1$.

Guessing matrices for $T_p(n, x)$

$p = 2$:

$$M_2(0) = \begin{bmatrix} 0 & 1 \\ -2x & 2x+1 \end{bmatrix} \quad M_2(1) = \begin{bmatrix} 2 & 0 \\ 2 & x \end{bmatrix}$$

$p = 3$:

$$M_3(0) = \begin{bmatrix} 0 & 1 \\ -3x & 3x+1 \end{bmatrix} \quad M_3(1) = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & 2x+\frac{1}{2} \end{bmatrix} \quad M_3(2) = \begin{bmatrix} 3 & 0 \\ 3x+3 & x \end{bmatrix}$$

$p = 5$:

$$M_5(0) = \begin{bmatrix} 0 & 1 \\ -5x & 5x+1 \end{bmatrix} \quad M_5(1) = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{5}{4} & 4x+\frac{3}{4} \end{bmatrix} \quad \cdots \quad M_5(4) = \begin{bmatrix} 5 & 0 \\ 15x+5 & x \end{bmatrix}$$

General p:

$$M_p(d) = \begin{bmatrix} \frac{dp}{p-1} & \frac{p-1-d}{p-1} \\ (d-1)px + \frac{dp}{p-1} & (p-d)x + \frac{p-1-d}{p-1} \end{bmatrix}.$$

But this matrix isn't unique... There are many bases.

Which basis is best?

Can we get integer coefficients?
Can we get non-negative integer coefficients? (allows a bijective proof)

For each 2×2 invertible matrix S with integer entries $\leq j$, compute

$$S^{-1} M_p(d) S.$$

$$T_p(n, x) = \begin{bmatrix} 1 & 0 \end{bmatrix} M_p(n_0)\, M_p(n_1) \cdots M_p(n_\ell) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Simplest matrix (maximizing monomial entries):

$$\begin{bmatrix} d+1 & p-d-1 \\ d\,x & (p-d)\,x \end{bmatrix}$$

Matrix product

Let

$$\color{red}{M_p(d)} := \begin{bmatrix} d+1 & p-d-1 \\ d\,x & (p-d)\,x \end{bmatrix}.$$

Theorem (Rowland 2018)

Write $n = n_\ell \cdots n_1 n_0$ in base p. Then

$$T_p(n,x) := \sum_{m=0}^{n} x^{\nu_p(\binom{n}{m})} = \begin{bmatrix} 1 & 0 \end{bmatrix} M_p(n_0)\, M_p(n_1) \,\cdots\, M_p(n_\ell) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Setting $x = 0$ gives $\theta_{p,0}(n) = (n_0 + 1) \cdots (n_\ell + 1)$ as a special case:

$$\begin{bmatrix} \theta_{p,0}(pn+d) \\ 0 \end{bmatrix} = \begin{bmatrix} d+1 & p-d-1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{p,0}(n) \\ 0 \end{bmatrix},$$

or simply

$$\theta_{p,0}(pn+d) = (d+1)\,\theta_{p,0}(n).$$

Comparing recurrences

Carlitz recurrence:

$$\theta_{p,\alpha}(pn+d) = (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1)$$
$$\psi_{p,\alpha}(pn+d) = \begin{cases} (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1) & \text{if } 0 \leq d \leq p-2 \\ p\psi_{p,\alpha-1}(n) & \text{if } d = p-1. \end{cases}$$

Carlitz has $\psi_{p,\alpha}(pn+d)$ on the left but $\psi_{p,\alpha-1}(n-1)$ on the right.

Recurrence leading to matrix product:

$$\theta_{p,\alpha}(pn+d) = (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1)$$
$$\psi_{p,\alpha}(pn+d-1) = d\theta_{p,\alpha}(n) + (p-d)\psi_{p,\alpha-1}(n-1).$$

$$M_p(d) = \begin{bmatrix} d+1 & p-d-1 \\ d\,x & (p-d)\,x \end{bmatrix}$$

Multinomial coefficients

For a k-tuple $\mathbf{m} = (m_1, m_2, \ldots, m_k)$ of non-negative integers, define

$$\text{total}\,\mathbf{m} := m_1 + m_2 + \cdots + m_k$$

and

$$\text{mult}\,\mathbf{m} := \frac{(\text{total}\,\mathbf{m})!}{m_1!\, m_2! \,\cdots\, m_k!}.$$

Theorem (Rowland 2018)

Let $k \geq 1$, and let $e = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{Z}^k$.
Write $n = n_\ell \cdots n_1 n_0$ in base p. Then

$$\sum_{\substack{\mathbf{m} \in \mathbb{N}^k \\ \text{total}\,\mathbf{m} = n}} x^{\nu_p(\text{mult}\,\mathbf{m})} = e\, M_{p,k}(n_0)\, M_{p,k}(n_1) \,\cdots\, M_{p,k}(n_\ell)\, e^\top.$$

$M_{p,k}(d)$ is a $k \times k$ matrix ...

Multinomial coefficients

Let $c_{p,k}(n)$ be the coefficient of x^n in $(1 + x + x^2 + \cdots + x^{p-1})^k$. $p = 5$:

						1						
				1	1	1	1	1				
		1	2	3	4	5	4	3	2	1		
1	3	6	10	15	18	19	18	15	10	6	3	1

For each $d \in \{0, \ldots, p-1\}$, let $M_{p,k}(d)$ be the $k \times k$ matrix whose (i, j) entry is $c_{p,k}(p\,(j-1) + d - (i-1))\,x^{i-1}$.

Example

Let $p = 5$ and $k = 3$; the matrices $M_{5,3}(0), \ldots, M_{5,3}(4)$ are

$$\begin{bmatrix} 1 & 18 & 6 \\ 0 & 15x & 10x \\ 0 & 10x^2 & 15x^2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 19 & 3 \\ x & 18x & 6x \\ 0 & 15x^2 & 10x^2 \end{bmatrix}, \quad \begin{bmatrix} 6 & 18 & 1 \\ 3x & 19x & 3x \\ x^2 & 18x^2 & 6x^2 \end{bmatrix},$$

$$\begin{bmatrix} 10 & 15 & 0 \\ 6x & 18x & x \\ 3x^2 & 19x^2 & 3x^2 \end{bmatrix}, \quad \begin{bmatrix} 15 & 10 & 0 \\ 10x & 15x & 0 \\ 6x^2 & 18x^2 & x^2 \end{bmatrix}.$$

Sketch of proof

Lemma

Let $n \geq 0$*.*
Let $k \geq 1$*.*
Let $0 \leq i \leq k-1$*.*
Let $d \in \{0, \dots, p-1\}$*.*
Let $\mathbf{m} \in \mathbb{N}^k$ *with* $\operatorname{total} \mathbf{m} = pn + d - i$*.*
Define $j = n - \operatorname{total} \lfloor \mathbf{m}/p \rfloor$*.*
Then $\operatorname{total}(\mathbf{m} \bmod p) = pj + d - i$*,* $0 \leq j \leq k-1$*, and*

$$\nu_p(\operatorname{mult} \mathbf{m}) + \nu_p\left(\frac{(pn+d)!}{(pn+d-i)!}\right) = \nu_p(\operatorname{mult} \lfloor \mathbf{m}/p \rfloor) + \nu_p\left(\frac{n!}{(n-j)!}\right) + j.$$

Sketch of proof

For $d \in \{0, \ldots, p-1\}$, $0 \leq i \leq k-1$, and $\alpha \geq 0$, show that

$$\beta(\mathbf{m}) := (\lfloor \mathbf{m}/p \rfloor, \mathbf{m} \bmod p)$$

is a bijection from the set

$$A = \left\{ \mathbf{m} \in \mathbb{N}^k : \operatorname{total} \mathbf{m} = pn + d - i \text{ and } \nu_p(\operatorname{mult} \mathbf{m}) = \alpha - \nu_p\left(\frac{(pn+d)!}{(pn+d-i)!} \right) \right\}$$

to the set

$$B = \bigcup_{j=0}^{k-1} \left(\left\{ \mathbf{c} \in \mathbb{N}^k : \operatorname{total} \mathbf{c} = n - j \text{ and } \nu_p(\operatorname{mult} \mathbf{c}) = \alpha - \nu_p\left(\frac{n!}{(n-j)!} \right) - j \right\} \times \left\{ \mathbf{d} \in \{0, \ldots, p-1\}^k : \operatorname{total} \mathbf{d} = pj + d - i \right\} \right).$$

The lemma implies that if $\mathbf{m} \in A$ then $\beta(\mathbf{m}) \in B$.

Unexplored territory

Do generalizations of binomial coefficients have analogous products?

- Fibonomial coefficients
- q-binomial coefficients
- Carlitz binomial coefficients
- word binomial coefficients $\binom{u}{v}$
- other hypergeometric terms $\binom{n}{m} = \frac{n!}{m!(n-m)!}$
- coefficients in other rational series $\binom{n+m}{m} = [x^n y^m]\frac{1}{1-x-y}$
- coefficients in $(1 + x + x^2 + \cdots + x^{p-1})^k$:

						1						
				1	1	1	1	1				
		1	2	3	4	5	4	3	2	1		
1	3	6	10	15	18	19	18	15	10	6	3	1