Binomial coefficients, valuations, and words

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Developments in Language Theory
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Valuations of binomial coefficients

Pascal's triangle:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

For this talk: p is a prime.

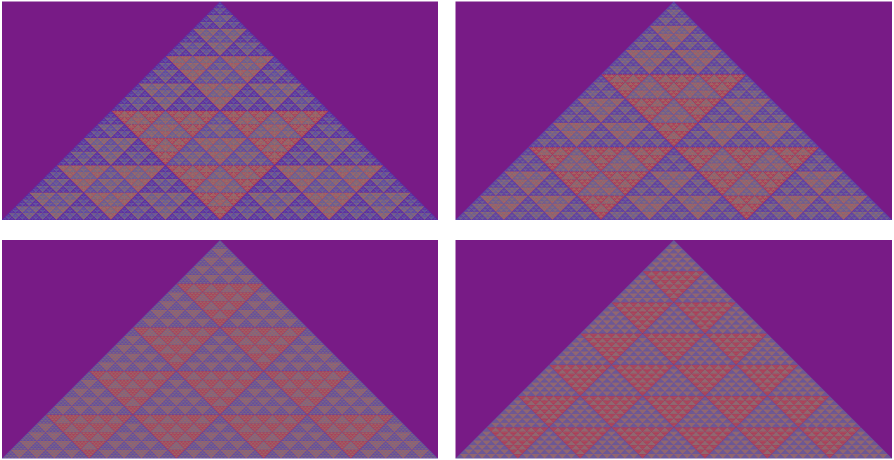
Let $\nu_p(n)$ denote the exponent of the highest power of p dividing n.
Example: $\nu_3(18) = 2$.

$$\nu_2(n): \quad 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, \ldots$$

Theorem (Kummer 1852)

$\nu_p(\binom{n}{m})$ = *number of carries involved in adding m to $n - m$ in base p.*

Valuations of binomial coefficients



Kummer's theorem

Theorem (Kummer)

$\nu_p(\binom{n}{m})$ = *number of carries involved in adding m to $n - m$ in base p.*

Proof: Use Legendre's formula

$$\nu_p(m!) = \frac{m - \sigma_p(m)}{p - 1},$$

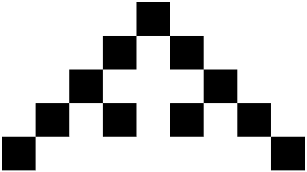
where $\sigma_p(m)$ is the sum of the base-p digits of m.

$$\begin{aligned}\nu_p\left(\binom{n}{m}\right) &= \nu_p\left(\frac{n!}{m!(n-m)!}\right) && (p \text{ is prime}) \\ &= \frac{n - \sigma_p(n)}{p-1} - \frac{m - \sigma_p(m)}{p-1} - \frac{n - m - \sigma_p(n-m)}{p-1} \\ &= \frac{-\sigma_p(n) + \sigma_p(m) + \sigma_p(n-m)}{p-1}.\end{aligned}$$

Odd binomial coefficients

Main theme: Arithmetic information about binomial coefficients reflects the base-p representations of integers.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$



Glaisher (1899) counted odd binomial coefficients:

$$1, 2, 2, 4, 2, 4, 4, 8, 2, 4, 4, 8, 4, 8, 8, 16, \ldots \qquad \theta_{2,0}(n) = 2^{|n|_1}$$

Definition

$\theta_{p,\alpha}(n) := \left|\{m : 0 \leq m \leq n \text{ and } \nu_p(\binom{n}{m}) = \alpha\}\right|$.

$|n|_w :=$ number of occurrences of w in the base-p representation of n.

Derivation from Kummer's theorem

Glaisher's result $\theta_{2,0}(n) = 2^{|n|_1}$ follows from Kummer's theorem.

Theorem (Kummer)

$\nu_p(\binom{n}{m}) =$ *number of carries involved in adding m to $n - m$ in base p.*

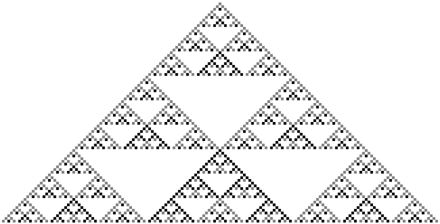
Example

$n = 25$, $\nu_2(\binom{25}{m}) = 0$.

$$\begin{aligned} n &= 25 = 11001_2 \\ m &= **00*_2 \end{aligned}$$

$\theta_{2,0}(25) = 2^{|25|_1} = 8$.

Binomial coefficients not divisible by p



Number of binomial coefficients with 3-adic valuation 0:

$$1, 2, 3, 2, 4, 6, 3, 6, 9, 2, 4, 6, 4, 8, 12, 6, \ldots \qquad \theta_{3,0}(n) = 2^{|n|_1} 3^{|n|_2}$$

Theorem (Fine 1947)

Write $n = n_\ell \cdots n_1 n_0$ in base p. Then

$$\theta_{p,0}(n) = (n_0 + 1) \cdots (n_\ell + 1) = 1^{|n|_0} 2^{|n|_1} 3^{|n|_2} \cdots p^{|n|_{p-1}}.$$

Prime powers?

Carlitz found a recurrence involving $\theta_{p,\alpha}(n)$ and a secondary quantity $\psi_{p,\alpha}(n) := \left|\{m : 0 \leq m \leq n \text{ and } \nu_p((m+1)\binom{n}{m}) = \alpha\}\right|$.

Theorem (Carlitz 1967)

$$\theta_{p,\alpha}(pn+d) = (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1)$$

$$\psi_{p,\alpha}(pn+d) = \begin{cases} (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1) & \text{if } 0 \leq d \leq p-2 \\ p\psi_{p,\alpha-1}(n) & \text{if } d = p-1. \end{cases}$$

Corollary (Carlitz)

Write $n = n_\ell \cdots n_1 n_0$ in base p. Then

$$\frac{\theta_{p,1}(n)}{\theta_{p,0}(n)} = \sum_{i=0}^{\ell-1} \frac{p-n_i-1}{n_i+1} \cdot \frac{n_{i+1}}{n_{i+1}+1}.$$

Expressions for $\frac{\theta_{p,\alpha}(n)}{\theta_{p,0}(n)}$ can be simpler than expressions for $\theta_{p,\alpha}(n)$.

Formulas for $\alpha = 1$

$\frac{\theta_{p,1}(n)}{\theta_{p,0}(n)}$ is a weighted sum of $|n|_w$ over $w \in \{0, \dots, p-1\}^*$ of length 2:

$$\theta_{2,1}(n) = 2^{|n|_1} \cdot \frac{1}{2}|n|_{10}$$

(Howard 1971; Davis–Webb 1989)

$$\theta_{3,1}(n) = 2^{|n|_1} 3^{|n|_2} \left(|n|_{10} + \frac{1}{4}|n|_{11} + \frac{4}{3}|n|_{20} + \frac{1}{3}|n|_{21}\right)$$

(Huard–Spearman–Williams 1997)

$$\begin{aligned}\theta_{5,1}(n) = 2^{|n|_1} 3^{|n|_2} 4^{|n|_3} 5^{|n|_4} \Big(& 2|n|_{10} + \frac{3}{4}|n|_{11} + \frac{1}{3}|n|_{12} + \frac{1}{8}|n|_{13} \\ & + \frac{8}{3}|n|_{20} + |n|_{21} + \frac{4}{9}|n|_{22} + \frac{1}{6}|n|_{23} \\ & + 3|n|_{30} + \frac{9}{8}|n|_{31} + \frac{1}{2}|n|_{32} + \frac{3}{16}|n|_{33} \\ & + \frac{16}{5}|n|_{40} + \frac{6}{5}|n|_{41} + \frac{8}{15}|n|_{42} + \frac{1}{5}|n|_{43}\Big)\end{aligned}$$

Formulas for $\alpha \geq 2$

Howard (1971) produced formulas for $\theta_{2,2}(n)$, $\theta_{2,3}(n)$, and $\theta_{2,4}(n)$.

$$\theta_{2,2}(n) = 2^{|n|_1} \left(-\frac{1}{8}|n|_{10} + |n|_{100} + \frac{1}{4}|n|_{110} + \frac{1}{8}|n|_{10}^2 \right)$$

(rediscovered by Huard–Spearman–Williams 1998)

$$\theta_{2,3}(n) = 2^{|n|_1} \left(\frac{1}{24}|n|_{10} - \frac{1}{2}|n|_{100} - \frac{1}{8}|n|_{110} + 2|n|_{1000} + \frac{1}{2}|n|_{1010} + \frac{1}{2}|n|_{1100} + \frac{1}{8}|n|_{1110} - \frac{1}{16}|n|_{10}^2 + \frac{1}{2}|n|_{10}|n|_{100} + \frac{1}{8}|n|_{10}|n|_{110} + \frac{1}{48}|n|_{10}^3 \right)$$

In studying the asymptotic behavior of $\sum_{n=0}^{N} \theta_{p,\alpha}(n)$, Barat and Grabner (2001) showed implicitly that $\frac{\theta_{p,\alpha}(n)}{\theta_{p,0}(n)}$ is a polynomial in $|n|_w$.

I worked out an algorithm for computing a polynomial expression.

Theorem (Rowland 2011)

$\frac{\theta_{p,\alpha}(n)}{\theta_{p,0}(n)}$ is equal to a polynomial of degree α in $|n|_w$ for words w satisfying $|w| \leq \alpha + 1$.

Coefficient of $|n|_{10}$

Spiegelhofer and Wallner produced a faster algorithm by developing a better understanding of the structure of this polynomial.

$$\theta_{2,1}(n) = 2^{|n|_1} \cdot \tfrac{1}{2}|n|_{10} \qquad \theta_{2,4}(n) = 2^{|n|_1}\left(-\tfrac{1}{64}|n|_{10} + \cdots\right)$$
$$\theta_{2,2}(n) = 2^{|n|_1}\left(-\tfrac{1}{8}|n|_{10} + \cdots\right) \qquad \theta_{2,5}(n) = 2^{|n|_1}\left(\tfrac{1}{160}|n|_{10} + \cdots\right)$$
$$\theta_{2,3}(n) = 2^{|n|_1}\left(\tfrac{1}{24}|n|_{10} + \cdots\right) \qquad \theta_{2,6}(n) = 2^{|n|_1}\left(-\tfrac{1}{384}|n|_{10} + \cdots\right)$$

These are the coefficients in $\log(1 + \frac{x}{2}) = \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{24}x^3 - \frac{1}{64}x^4 + \cdots$.

Polynomials in $|n|_w$ aren't unique: $|n|_{10} = |n|_{010} + |n|_{110}$.

A word $w \in \{0, \ldots, p-1\}^*$ is admissible if $|w| \geq 2$ and w isn't of the form $0v$ or $v(p-1)$.

Spiegelhofer–Wallner:
$\frac{\theta_{p,\alpha}(n)}{\theta_{p,0}(n)}$ is a *unique* polynomial function of $\{|n|_w : w \text{ admissible}\}$.
The coefficient of a monomial can be read off from a power series.

Generating function

Define

$$\color{red}{T_p(n,x)} := \sum_{m=0}^{n} x^{\nu_p(\binom{n}{m})} = \sum_{\alpha \geq 0} \theta_{p,\alpha}(n) x^\alpha.$$

In particular, $T_p(n,0) = \theta_{p,0}(n)$.

$p = 2$:

n	$T_2(n,x)$
0	1
1	2
2	$x + 2$
3	4
4	$2x^2 + x + 2$
5	$2x + 4$
6	$x^2 + 2x + 4$
7	8

n	$T_2(n,x)$
8	$4x^3 + 2x^2 + x + 2$
9	$4x^2 + 2x + 4$
10	$2x^3 + x^2 + 4x + 4$
11	$4x + 8$
12	$2x^3 + 5x^2 + 2x + 4$
13	$2x^2 + 4x + 8$
14	$x^3 + 2x^2 + 4x + 8$
15	16

Coefficients

Theorem (Spiegelhofer–Wallner 2016)

Let $w_1, \ldots, w_m$ be admissible words. The coefficient of $|n|_{w_1}^{k_1} \cdots |n|_{w_m}^{k_m}$ in $\frac{\theta_{p,\alpha}(n)}{\theta_{p,0}(n)}$ is the coefficient of x^α in the power series for

$$\frac{1}{k_1!}(\log r_p(w_1, x))^{k_1} \cdots \frac{1}{k_m!}(\log r_p(w_m, x))^{k_m}.$$

…where $r_p(w, x)$ is a rational function defined by

$$r_p(w, x) := \frac{\overline{T}_p(w, x)\overline{T}_p(w_{LR}, x)}{\overline{T}_p(w_R, x)\overline{T}_p(w_L, x)}, \qquad \overline{T}_p(w, x) := \frac{T_p(\mathrm{val}_p(w), x)}{\theta_{p,0}(\mathrm{val}_p(w))},$$

$\mathrm{val}_p(w)$ is the integer obtained by reading w in base p, and the left and right truncations of a word are defined by

$$\begin{aligned} \epsilon_L &= \epsilon & (c0^\ell)_L &= \epsilon & (c0^\ell w)_L &= w \\ \epsilon_R &= \epsilon & c_R &= \epsilon & (wd)_R &= w \end{aligned}$$

for $\ell \geq 0$, $c \in \{1, \ldots, p-1\}$, and $d \in \{0, \ldots, p-1\}$.

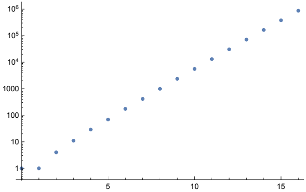
Sketch of proof

$$\begin{aligned}
\frac{\theta_{p,\alpha}(n)}{\theta_{p,0}(n)} &= [x^\alpha]\frac{T_p(n,x)}{\theta_{p,0}(n)} && \text{(by definition of } T\text{)} \\
&= [x^\alpha] \prod_{\substack{w \text{ admissible} \\ |w| \leq \alpha+1}} r_p(w,x)^{|n|_w} && \text{(main lemma)} \\
&= [x^\alpha] \prod_{\substack{w \text{ admissible} \\ |w| \leq \alpha+1}} \exp(|n|_w \log r_p(w,x)) \\
&= [x^\alpha] \prod_{\substack{w \text{ admissible} \\ |w| \leq \alpha+1}} \sum_{k \geq 0} |n|_w^k \frac{(\log r_p(w,x))^k}{k!}
\end{aligned}$$

Formula size

$p = 2$: Number of nonzero monomials in $\frac{\theta_{2,\alpha}(n)}{2^{|n|_1}}$ for $\alpha = 0, 1, 2, \ldots$:

$1, 1, 4, 11, 29, 69, 174, 413, 995, 2364, 5581,$
$13082, 30600, 71111, 164660, 379682, 872749, \ldots$ (A275012)



$$\Theta\left(\alpha^{-3/4} p^{\alpha} e^{2(p-1)\sqrt{\alpha/p}}\right)$$

So we can compute a formula for $\theta_{p,\alpha}(n)$ for fixed p, α.
But is there a (product?) formula for $\theta_{p,\alpha}(n)$ for symbolic p, α?

There is a product formula for $T_p(n, x) := \sum_{m=0}^{n} x^{\nu_p(\binom{n}{m})}$!

Matrix product

Let

$$\color{red}{M_p(d)} := \begin{bmatrix} d+1 & p-d-1 \\ d\,x & (p-d)\,x \end{bmatrix}.$$

Theorem (Rowland 2017)

Write $n = n_\ell \cdots n_1 n_0$ in base p. Then

$$T_p(n,x) := \sum_{m=0}^{n} x^{\nu_p(\binom{n}{m})} = \begin{bmatrix} 1 & 0 \end{bmatrix} M_p(n_0)\, M_p(n_1) \,\cdots\, M_p(n_\ell) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Setting $x = 0$ gives $\theta_{p,0}(n) = (n_0 + 1) \cdots (n_\ell + 1)$ as a special case:

$$\begin{bmatrix} \theta_{p,0}(pn+d) \\ 0 \end{bmatrix} = \begin{bmatrix} d+1 & p-d-1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{p,0}(n) \\ 0 \end{bmatrix},$$

or simply

$$\theta_{p,0}(pn+d) = (d+1)\,\theta_{p,0}(n).$$

Comparing recurrences

Carlitz recurrence:

$$\theta_{p,\alpha}(pn+d) = (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1)$$
$$\psi_{p,\alpha}(pn+d) = \begin{cases} (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1) & \text{if } 0 \le d \le p-2 \\ p\psi_{p,\alpha-1}(n) & \text{if } d = p-1. \end{cases}$$

We have $\psi_{p,\alpha-1}(n-1)$ on the right but $\psi_{p,\alpha}(pn+d)$ on the left.

Recurrence leading to matrix product:

$$\begin{aligned} \theta_{p,\alpha}(pn+d) &= (d+1)\theta_{p,\alpha}(n) + (p-d-1)\psi_{p,\alpha-1}(n-1) \\ \psi_{p,\alpha}(pn+d-1) &= d\theta_{p,\alpha}(n) + (p-d)\psi_{p,\alpha-1}(n-1). \end{aligned}$$

$$M_p(d) = \begin{bmatrix} d+1 & p-d-1 \\ d\,x & (p-d)\,x \end{bmatrix}$$

Multinomial coefficients

For a k-tuple $\mathbf{m} = (m_1, m_2, \ldots, m_k)$ of non-negative integers, define

$$\operatorname{total} \mathbf{m} := m_1 + m_2 + \cdots + m_k$$

and

$$\operatorname{mult} \mathbf{m} := \frac{(\operatorname{total} \mathbf{m})!}{m_1!\, m_2! \,\cdots\, m_k!}.$$

Theorem (Rowland 2017)

Let $k \geq 1$, and let $e = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{Z}^k$.
Write $n = n_\ell \cdots n_1 n_0$ in base p. Then

$$\sum_{\substack{\mathbf{m} \in \mathbb{N}^k \\ \operatorname{total} \mathbf{m} = n}} x^{\nu_p(\operatorname{mult} \mathbf{m})} = e\, M_{p,k}(n_0)\, M_{p,k}(n_1) \,\cdots\, M_{p,k}(n_\ell)\, e^\top.$$

$M_{p,k}(d)$ is a $k \times k$ matrix ...

Multinomial coefficients

Let $c_{p,k}(n)$ be the coefficient of x^n in $(1 + x + x^2 + \cdots + x^{p-1})^k$. $p = 5$:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & 1 & & & & \\
 & & 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 & & \\
1 & 3 & 6 & 10 & 15 & 18 & 19 & 18 & 15 & 10 & 6 & 3 & 1
\end{array}$$

For each $d \in \{0, \ldots, p-1\}$, let $M_{p,k}(d)$ be the $k \times k$ matrix whose (i, j) entry is $c_{p,k}(p\,(j-1) + d - (i-1))\, x^{i-1}$.

Example

Let $p = 5$ and $k = 3$; the matrices $M_{5,3}(0), \ldots, M_{5,3}(4)$ are

$$\begin{bmatrix} 1 & 18 & 6 \\ 0 & 15x & 10x \\ 0 & 10x^2 & 15x^2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 19 & 3 \\ x & 18x & 6x \\ 0 & 15x^2 & 10x^2 \end{bmatrix}, \quad \begin{bmatrix} 6 & 18 & 1 \\ 3x & 19x & 3x \\ x^2 & 18x^2 & 6x^2 \end{bmatrix},$$
$$\begin{bmatrix} 10 & 15 & 0 \\ 6x & 18x & x \\ 3x^2 & 19x^2 & 3x^2 \end{bmatrix}, \quad \begin{bmatrix} 15 & 10 & 0 \\ 10x & 15x & 0 \\ 6x^2 & 18x^2 & x^2 \end{bmatrix}.$$

p-regularity

One can guess Fine's theorem by factoring integers.

$$\theta_{p,0}(n) = (n_0 + 1) \cdots (n_\ell + 1) = 1^{|n|_0} 2^{|n|_1} 3^{|n|_2} \cdots p^{|n|_{p-1}}.$$

How to guess a matrix product?

Definition

A sequence $s(n)_{n \geq 0}$ is p-regular if the vector space generated by

$$\{s(p^e n + i)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq i \leq p^e - 1\}$$

is finite-dimensional.

Examples of p-regular sequences:

- $\nu_p(n)$
- $|n|_w$
- p-automatic sequences
- polynomial and quasi-polynomial sequences
- sums and products of p-regular sequences

Characterizations of p-regularity

Allouche & Shallit 1992:

- $\langle \{ s(p^e n + i)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq i \leq p^e - 1 \} \rangle$ is finite-dimensional
- $s(n)$ is determined by finitely many linear recurrences in $s(p^e n + i)$ (along with finitely many initial conditions)
- matrix product $s(n) = \lambda \, M(n_0) \, M(n_1) \, \cdots \, M(n_\ell) \, \kappa$
- generating function in p non-commuting variables is rational

Analogous characterizations of constant-recursive sequences:

- $\langle \{ s(n + i)_{n \geq 0} : i \geq 0 \} \rangle$ is finite-dimensional
- $s(n)$ is determined by a linear recurrence in $s(n + i)$ (along with finitely many initial conditions)
- matrix product $s(n) = \lambda \, M^n \, \kappa$
- generating function (in one variable) is rational

Guessing a 2-regular sequence

$s(n) = \theta_{2,1}(n) =$ number of binomial coefficients $\binom{n}{m}$ with $\nu_2(\binom{n}{m}) = 1$:

$s(n)$:	$0, 0, 1, 0, 1, 2, 2, 0, \dots$	basis element!
$s(2n+0)$:	$0, 1, 1, 2, 1, 4, 2, 4, \dots$	basis element!
$s(2n+1)$:	$0, 0, 2, 0, 2, 4, 4, 0, \dots$	$= 2s(n)$
$s(4n+0)$:	$0, 1, 1, 2, 1, 4, 2, 4, \dots$	$= s(2n)$
$s(4n+2)$:	$1, 2, 4, 4, 4, 8, 8, 8, \dots$	basis element!
$s(8n+2)$:	$1, 4, 4, 8, 4, 12, 8, 16, \dots$	$= 2s(2n) + s(4n+2) - 2s(n)$
$s(8n+6)$:	$2, 4, 8, 8, 8, 16, 16, 16, \dots$	$= 2s(4n+2)$

Convert the recurrence to matrices:

$$\begin{bmatrix} s(2n) \\ s(4n) \\ s(8n+2) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix}$$

$$\begin{bmatrix} s(2n+1) \\ s(4n+2) \\ s(8n+6) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} s(n) \\ s(2n) \\ s(4n+2) \end{bmatrix}$$

An implementation in *Mathematica*

IntegerSequences is available from

`https://people.hofstra.edu/Eric_Rowland/packages.html`

```
In[1]:= Import["https://people.hofstra.edu/Eric_Rowland/packages/IntegerSequences.m"]
```

In[2]:= $\texttt{Table}\left[\sum_{m=0}^{n} x^{\texttt{IntegerExponent[Binomial[n,m],2]}}, \{n, 0, 10\}\right]$

Out[2]= $\{1, 2, 2+x, 4, 2+x+2x^2, 4+2x, 4+2x+x^2, 8, 2+x+2x^2+4x^3, 4+2x+4x^2, 4+4x+x^2+2x^3\}$

```
In[3]:= FindRegularSequenceFunction[%, 2] // RegularSequenceMatrixForm
```

Out[3]= $\texttt{RegularSequence}\left[\{1, 0\}, \left\{\begin{pmatrix} 0 & 1 \\ -2x & 1+2x \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 2 & x \end{pmatrix}\right\}, \{1, 1\}\right]$

Guessing matrices for $T_p(n, x)$

$p = 2$:

$$M_2(0) = \begin{bmatrix} 0 & 1 \\ -2x & 2x+1 \end{bmatrix} \quad M_2(1) = \begin{bmatrix} 2 & 0 \\ 2 & x \end{bmatrix}$$

$p = 3$:

$$M_3(0) = \begin{bmatrix} 0 & 1 \\ -3x & 3x+1 \end{bmatrix} \quad M_3(1) = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & 2x+\frac{1}{2} \end{bmatrix} \quad M_3(2) = \begin{bmatrix} 3 & 0 \\ 3x+3 & x \end{bmatrix}$$

$p = 5$:

$$M_5(0) = \begin{bmatrix} 0 & 1 \\ -5x & 5x+1 \end{bmatrix} \quad M_5(1) = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{5}{4} & 4x+\frac{3}{4} \end{bmatrix} \quad \cdots \quad M_5(4) = \begin{bmatrix} 5 & 0 \\ 15x+5 & x \end{bmatrix}$$

General p:

$$M_p(d) = \begin{bmatrix} \frac{dp}{p-1} & \frac{p-1-d}{p-1} \\ (d-1)px + \frac{dp}{p-1} & (p-d)x + \frac{p-1-d}{p-1} \end{bmatrix}.$$

But this choice of matrices isn't unique... There are many bases.

Which basis is best?

Can we get integer coefficients?
Can we get non-negative integer coefficients? (allows a bijective proof)

For each 2×2 invertible matrix S with integer entries $\leq j$, compute

$$S^{-1} M_p(d) S.$$

$$T_p(n, x) = \begin{bmatrix} 1 & 0 \end{bmatrix} M_p(n_0) \, M_p(n_1) \, \cdots \, M_p(n_\ell) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Matrix with fewest monomials:

$$M_p(d) = \begin{bmatrix} d+1 & p-d-1 \\ d\,x & (p-d)\,x \end{bmatrix}$$

Sketch of proof

Lemma

Let $n \geq 0$.
Let $k \geq 1$.
Let $0 \leq i \leq k - 1$.
Let $d \in \{0, \dots, p - 1\}$.
Let $\mathbf{m} \in \mathbb{N}^k$ *with* $\text{total}\, \mathbf{m} = pn + d - i$.
Define $j = n - \text{total}\lfloor \mathbf{m}/p \rfloor$.
Then $\text{total}(\mathbf{m} \bmod p) = pj + d - i$, $0 \leq j \leq k - 1$, *and*

$$\nu_p\left(\frac{(pn+d)!}{(pn+d-i)!}\right) + \nu_p(\text{mult}\, \mathbf{m}) = \nu_p\left(\frac{n!}{(n-j)!}\right) + \nu_p(\text{mult}\lfloor \mathbf{m}/p \rfloor) + j.$$

Sketch of proof

For $d \in \{0, \dots, p-1\}$, $0 \leq i \leq k-1$, and $\alpha \geq 0$, show that

$$\beta(\mathbf{m}) := (\lfloor \mathbf{m}/p \rfloor, \mathbf{m} \bmod p)$$

is a bijection from the set

$$A = \left\{ \mathbf{m} \in \mathbb{N}^k : \operatorname{total} \mathbf{m} = pn + d - i \text{ and } \nu_p(\operatorname{mult} \mathbf{m}) = \alpha - \nu_p\left(\frac{(pn+d)!}{(pn+d-i)!} \right) \right\}$$

to the set

$$B = \bigcup_{j=0}^{k-1} \left(\left\{ \mathbf{c} \in \mathbb{N}^k : \operatorname{total} \mathbf{c} = n - j \text{ and } \nu_p(\operatorname{mult} \mathbf{c}) = \alpha - \nu_p\left(\frac{n!}{(n-j)!} \right) - j \right\} \times \left\{ \mathbf{d} \in \{0, \dots, p-1\}^k : \operatorname{total} \mathbf{d} = pj + d - i \right\} \right).$$

The previous lemma implies that if $\mathbf{m} \in A$ then $\beta(\mathbf{m}) \in B$.

Unexplored territory

Do generalizations of binomial coefficients have analogous products?

- Fibonomial coefficients
- q-binomial coefficients
- Carlitz binomial coefficients
- word binomial coefficients $\binom{u}{v}$
- other hypergeometric terms $\qquad \binom{n}{m} = \frac{n!}{m!(n-m)!}$
- coefficients in other rational series $\qquad \binom{n+m}{m} = [x^n y^m] \frac{1}{1-x-y}$
- coefficients in $(1 + x + x^2 + \cdots + x^{p-1})^k$:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & 1 & & & & \\
 & & 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 & & \\
1 & 3 & 6 & 10 & 15 & 18 & 19 & 18 & 15 & 10 & 6 & 3 & 1
\end{array}
$$