Avoiding repetitions in sequences of integers

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Periodic sequences

$$000000000000000000000000000000000000000000\cdots$$
$$011011011011011011011011011011011011011011\cdots$$
$$020302030203020302030203020302030203020302\cdots$$

Periodic sequences are simple!

What’s a more interesting (but related) class of sequences?
We could…

- Maximize simplicity:
  What are the “simplest” non-periodic sequences?
- Maximize non-periodicity:
  How “non-periodic” can a sequence be?

A square is a nonempty word of the form ww. Example: hotshots

Squares on a 2-letter alphabet

Axel Thue (1863–1922)

Are squares avoidable on a 2-letter alphabet?
Are there arbitrarily long square-free words on $\{0, 1\}$?

Choose an order on $\{0, 1\}$ and try to construct one:

$$010\boxtimes$$

Squares on a 3-letter alphabet

Are squares avoidable on $\{0, 1, 2\}$?

$$0102012021012010201202102010210120102012021 0\cdots$$

Theorem (Thue 1906)

There exist arbitrarily long square-free words on 3 *letters.*

The backtracking algorithm builds the lexicographically least sequence.

Open problem (Allouche–Shallit, *Automatic Sequences* §1.10)

Characterize the lex. least square-free sequence on $\{0, 1, 2\}$.

Infinite alphabet

On an infinite alphabet, the backtracking algorithm doesn't backtrack.

Are squares avoidable on $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\}$? Yes.

$$0102010301020104010201030102010 5 \cdots$$

Theorem (Guay-Paquet–Shallit 2009)

Let $\varphi(n) = 0\,(n+1)$.
The lexicographically least square-free sequence on $\mathbb{Z}_{\geq 0}$ is $\varphi^\infty(0)$.

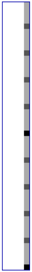
$$\begin{aligned}
\varphi(0) &= 01 \\
\varphi^2(0) &= 0102 \\
\varphi^3(0) &= 01020103 \\
&\vdots \\
\varphi^\infty(0) &= 01020103010201040102010301020105 \cdots
\end{aligned}$$

Integer powers

More generally, let $a \geq 2$. Let $\varphi(n) = 0^{a-1}(n+1)$.
The lexicographically least a-power-free sequence on $\mathbb{Z}_{\geq 0}$ is $\varphi^\infty(0)$.

$$\mathbf{s}_5 = 00001000010000100001000020000100001 \cdots$$

$$\begin{aligned} \mathbf{s}_5 = {} & 00001 \\ & 00001 \\ & 00001 \\ & 00001 \\ & 00002 \\ & 00001 \\ & \vdots \end{aligned}$$



$$\begin{aligned} s(5n+0) &= 0 \\ s(5n+1) &= 0 \\ s(5n+2) &= 0 \\ s(5n+3) &= 0 \\ s(5n+4) &= s(n)+1 \end{aligned}$$

$\mathbf{s}_5$ satisfies a recurrence reflecting the base-5 representation of n.
Such a sequence is called 5-regular.

Fractional powers

$011101 = (0111)^{3/2}$ is a $\frac{3}{2}$-power. decade
If $|x| = |y| = |z|$, then $xyzxyzx = (xyz)^{7/3}$ is a $\frac{7}{3}$-power. alfalfa

Definition

A word w is an $\frac{a}{b}$-power if

$$w = v^e x$$

where $e \geq 0$ is an integer, x is a prefix of v, and $\frac{|w|}{|v|} = \frac{a}{b}$.

Notation

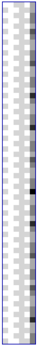
For $\frac{a}{b} > 1$, let $\mathbf{s}_{a/b}$ be the lex. least $\frac{a}{b}$-power-free sequence on $\mathbb{Z}_{\geq 0}$.

We assume $\gcd(a, b) = 1$ from now on.

Avoiding $3/2$-powers

$$\mathbf{s}_{3/2} = 00110210011200110310011300110210011400110 3\cdots$$

$$\begin{aligned}\mathbf{s}_{3/2} = {} & 001102 \\ & 100112 \\ & 001103 \\ & 100113 \\ & 001102 \\ & 100114 \\ & 001103 \\ & 100112 \\ & \vdots\end{aligned}$$



$$s(6n+5) = s(n) + 2$$

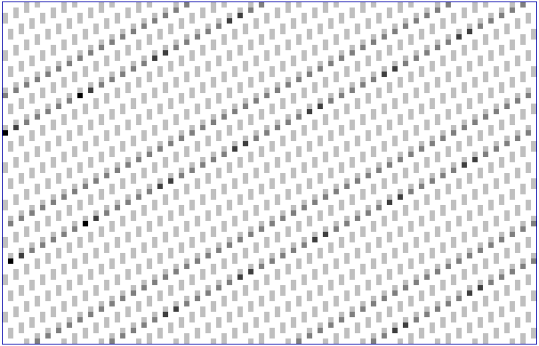
Theorem (Rowland–Shallit 2012)

The sequence $\mathbf{s}_{3/2}$ is 6-regular.

Why 6?

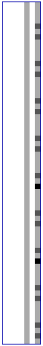
$\mathbf{s}_{5/3}$ wrapped into 100 columns

$\mathbf{s}_{5/3} = 0000101000010100001010000101000010200001 01\cdots$



$\mathbf{s}_{5/3}$ wrapped into 7 columns

$$\mathbf{s}_{5/3} = 00001010000101000010100001010000102000101\cdots$$

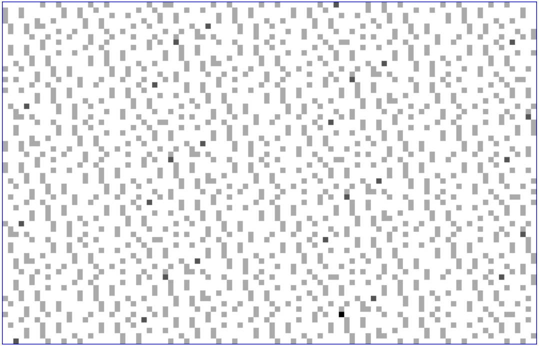


Theorem

$\mathbf{s}_{5/3} = \varphi^{\infty}(0)$, *where* $\varphi(n) = 000010(n+1)$ *is a 7-uniform morphism.*

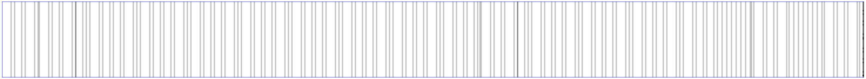
$\mathbf{s}_{8/5}$ wrapped into 100 columns

$$\mathbf{s}_{8/5} = 000000010010000001001000000001001100000000100\cdots$$



$\mathbf{s}_{8/5}$ wrapped into 733 columns

$$\mathbf{s}_{8/5} = 00000001001000000100100000000100110000000010000000100\cdots$$



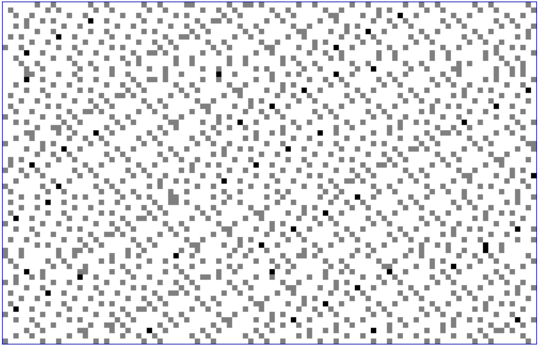
Theorem

$\mathbf{s}_{8/5} = \varphi^{\infty}(0)$ for the 733-uniform morphism

$$\begin{aligned}\varphi(n) = {} & 000000010010000001001000000001001100000000100100000010010000000010020000 \\ & 0100100100000000100100000010010000001001000000001001001000000000100100000 \\ & 100100000010010000000010010010000000010010000001001000000100100000000 1001 \\ & 0010000000010010000001001000000100100000000100100100000000100100000010010 \\ & 0000100100000000100100100000000100100000010010000001001000000000100100100 \\ & 000001001000000100100000010010000000010010010000000010010000001001000001 \\ & 0010110000000010010000001001000000001002000000100100100000000100100000010 \\ & 0100000010010000000010010010000000010010000001001000000100100000000100100 \\ & 1000000001001000000100100000010010000000010010010000000010010000001001000 \\ & 0010010001000100010001000100010001101000000001001000000100100000000101 \\ & 000100010001000100010001000101000000001001000000100100000000010100(n+2).\end{aligned}$$

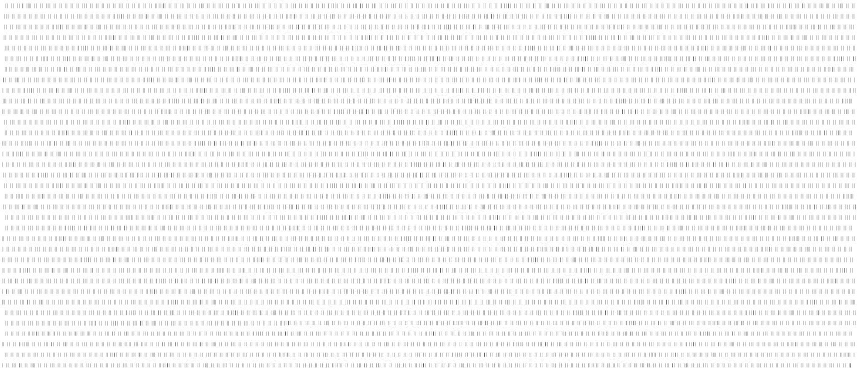
$\mathbf{s}_{7/4}$ wrapped into 100 columns

$$\mathbf{s}_{7/4} = 000000100100000010010000001001000011000000 \cdots$$



$\mathbf{s}_{7/4}$ wrapped into 50847 columns

$$\mathbf{s}_{7/4} = 000000100100000010010000001001000011000000\cdots$$

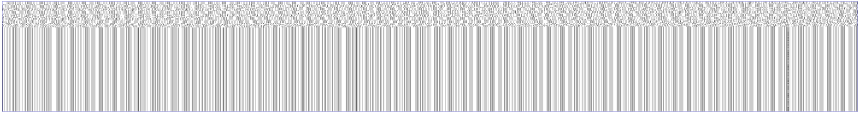


Theorem

$\mathbf{s}_{7/4} = \varphi^\infty(0)$ *for some* 50847*-uniform morphism* $\varphi(n) = u\,(n+2)$.

$\mathbf{s}_{6/5}$ wrapped into 1001 columns

$$\mathbf{s}_{6/5} = 00000111110202020101110100020212021011 0010\cdots$$



Introduce a new letter $0'$.
Let $\tau(0') = 0$ and $\tau(n) = n$ for $n \in \mathbb{Z}_{\geq 0}$.

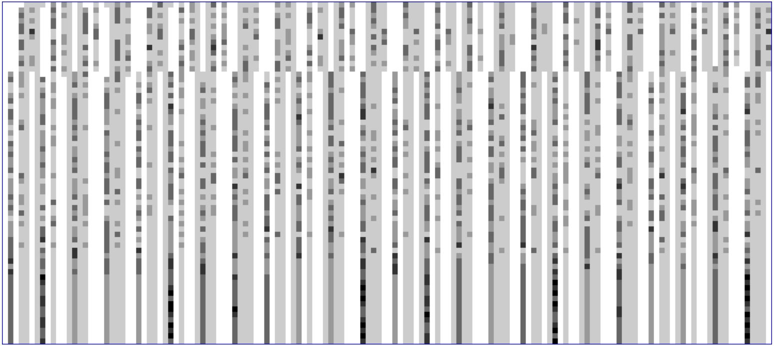
Theorem

There exist words u, v of lengths $|u| = 1001 - 1$ and $|v| = 29949$ such that $\mathbf{s}_{6/5} = \tau(\varphi^{\infty}(0'))$, where

$$\varphi(n) = \begin{cases} v\,\varphi(0) & \text{if } n = 0' \\ u\,(n+3) & \text{if } n \geq 0. \end{cases}$$

$\mathbf{s}_{5/4}$ wrapped into 144 columns

$$\mathbf{s}_{5/4} = 000011110202101001011212000013110102101302\cdots$$



We don't know the structure of $\mathbf{s}_{5/4}$.

Catalogue

For many sequences $\mathbf{s}_{a/b}$, there is a related k-uniform morphism.
A k-uniform morphism generates a k-regular sequence.

$\frac{a}{b}$	k
$\frac{3}{2}$	6
$\frac{5}{3}$	7
$\frac{8}{5}$	733
$\frac{7}{4}$	50847
$\frac{6}{5}$	1001
$\frac{5}{4}$	?

Question

Is every $\mathbf{s}_{a/b}$ k-regular for some k? How is k related to $\frac{a}{b}$?