Avoiding fractional powers over the natural numbers

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Squares and overlaps

A square is a nonempty word of the form *ww*.

An overlap is a word of the form *wwc*, where *c* is the first letter of *w*.

Squares are unavoidable on a binary alphabet:

$$010\boxed{?}$$

But Thue showed overlaps are avoidable:

$$\varphi^{\omega}(0) = 01101001100101101001011001101001 \cdots$$

is overlap-free, where $\varphi(0) = 01$ and $\varphi(1) = 10$.

Lexicographically least words

What is the lex. least infinite word avoiding a pattern?

On a binary alphabet, the lex. least overlap-free word is

$$001001\varphi^{\omega}(1) = 001001100101100110100 1\cdots.$$

Open problem (Allouche–Shallit, *Automatic Sequences* §1.10)

Characterize the lex. least square-free word over $\{0, 1, 2\}$.

$$010201202101201020120210201021012010201202 10\cdots$$

Infinite alphabet

Guay-Paquet–Shallit (2009):

The lex. least square-free word on $\mathbb{Z}_{\geq 0}$ is

$$\mathbf{w}_2 = \varphi^{\omega}(0) = 010201030102010401020103010201050 \cdots ,$$

where φ is the 2-uniform morphism $\varphi(n) = 0(n+1)$.

The lex. least 5-power-free word on $\mathbb{Z}_{\geq 0}$ is

$$\mathbf{w}_5 = \varphi^{\omega}(0) = 000010000100001000010000200001000001 \cdots ,$$

where $\varphi(n) = 0000(n+1)$.

The lex. least overlap-free word on $\mathbb{Z}_{\geq 0}$ is also generated by a (non-uniform) morphism.

Fractional powers

Definition

A word w is an $\frac{a}{b}$-power if

$$w = v^e x$$

where $e \geq 0$ is an integer, x is a prefix of v, and $\frac{|w|}{|v|} = \frac{a}{b}$.

If $|x| = |y| = |z|$, then $xyzxyzx = (xyz)^{7/3}$ is a $\frac{7}{3}$-power.
$011101 = (0111)^{3/2}$ is a $\frac{3}{2}$-power.

Notation

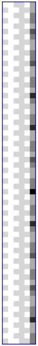
For $\frac{a}{b} > 1$, let $\mathbf{w}_{a/b}$ be the lex. least $\frac{a}{b}$-power-free word on $\mathbb{Z}_{\geq 0}$.

(We assume $\gcd(a, b) = 1$ from now on.)

Avoiding $3/2$-powers

$$\mathbf{w}_{3/2} = 001102100112001103100113001102100114001103\cdots$$

$$\begin{aligned}\mathbf{w}_{3/2} = \;&001102\\ &100112\\ &001103\\ &100113\\ &001102\\ &100114\\ &001103\\ &100112\\ &\vdots\end{aligned}$$



Theorem (Rowland–Shallit 2012)

$\mathbf{w}_{3/2}$ *is a* 6*-regular sequence and generated by a* 6*-uniform morphism.*

Why 6?

The interval $\frac{a}{b} \geq 2$

$$\mathbf{w}_{5/2} = 000010000100001000010000200001000010000100\cdots = \mathbf{w}_5$$

Theorem

If $\frac{a}{b} \geq 2$, then $\mathbf{w}_{a/b} = \mathbf{w}_a$.

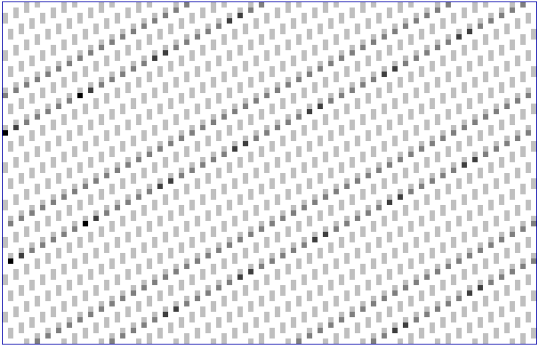
Proof (one direction).

The a-power $v^a = (v^b)^{a/b}$ is also an $\frac{a}{b}$-power.
So $\mathbf{w}_{a/b}$ is a-power-free. Thus $\mathbf{w}_a \leq \mathbf{w}_{a/b}$ lexicographically. $\square$

Therefore it suffices to consider $1 < \frac{a}{b} < 2$.

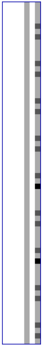
$\mathbf{w}_{5/3}$ wrapped into 100 columns

$$\mathbf{w}_{5/3} = 000010100001010000101000010100001020000101\cdots$$



$\mathbf{w}_{5/3}$ wrapped into 7 columns

$$\mathbf{w}_{5/3} = 000010100001010000101000010100001020000101 \cdots$$

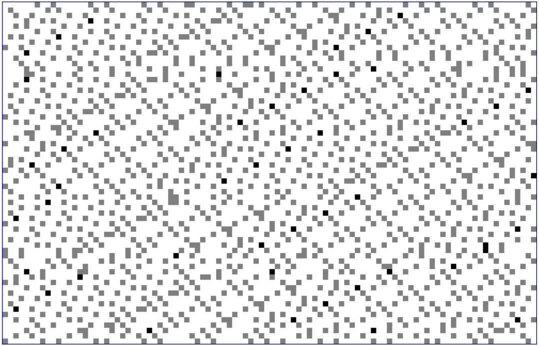


Theorem

$\mathbf{w}_{5/3} = \varphi^{\omega}(0)$, *where* $\varphi(n) = 000010(n+1)$ *is a* 7*-uniform morphism.*

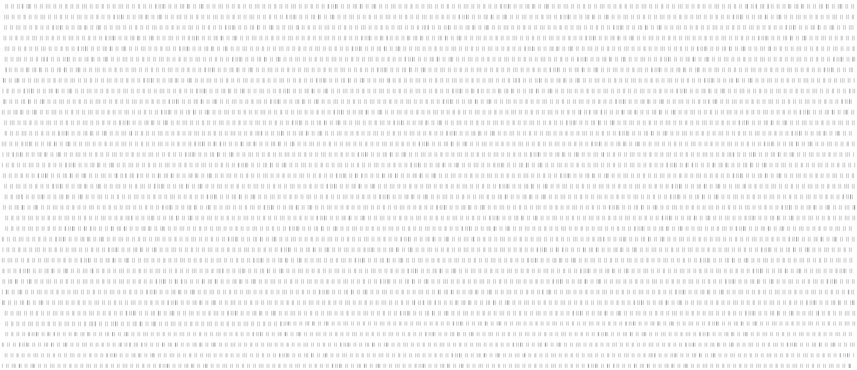
$\mathbf{w}_{7/4}$ wrapped into 100 columns

$$\mathbf{w}_{7/4} = 000000100100000010010000000100100001100000\cdots$$



$\mathbf{w}_{7/4}$ wrapped into 50847 columns

$$\mathbf{w}_{7/4} = 0000001001000000100100000010010000011000000 \cdots$$

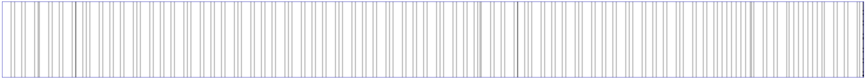


Theorem

$\mathbf{w}_{7/4} = \varphi^{\omega}(0)$ *for some* 50847*-uniform morphism* $\varphi(n) = u\,(n+2)$.

$\mathbf{w}_{8/5}$ wrapped into 733 columns

$$\mathbf{w}_{8/5} = 000000010010000001001000000001001100000000100 \cdots$$



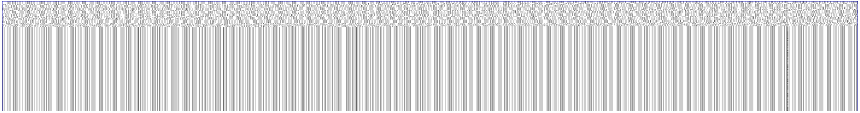
Theorem

$\mathbf{w}_{8/5} = \varphi^{\omega}(0)$ *for the 733-uniform morphism*

$$\begin{aligned}
\varphi(n) = {} & 0000000100100000010010000000010011000000001001000000100100000000010020000 \\
& 0100100100000000100100000010010000001001000000001001001000000001001000000 \\
& 1001000000100100000000100100100000000100100000010010000001001000000001001 \\
& 0010000000010010000001001000000100100000000100100100000000100100000010010 \\
& 0000100100000000100100100000000100100000010010000001001000000001001000100 \\
& 0000010010000001001000000100100000000100100100000000100100000010010000001 \\
& 0010110000000010010000001001000000001002000000100100100000000100100000010 \\
& 0100000010010000000010010010000000010010000001001000000100100000000100100 \\
& 1000000010010000001001000000100100000000100100100000000100100000010010000 \\
& 0010010001000100010001000100010001101000000001001000000100100000000101 \\
& 0001000100010001000100010001010000000010010000001001000000000010100(n+2).
\end{aligned}$$

$\mathbf{w}_{6/5}$ wrapped into 1001 columns

$$\mathbf{w}_{6/5} = 000001111102020201011101000202120210110010\cdots$$



There is a transient region.

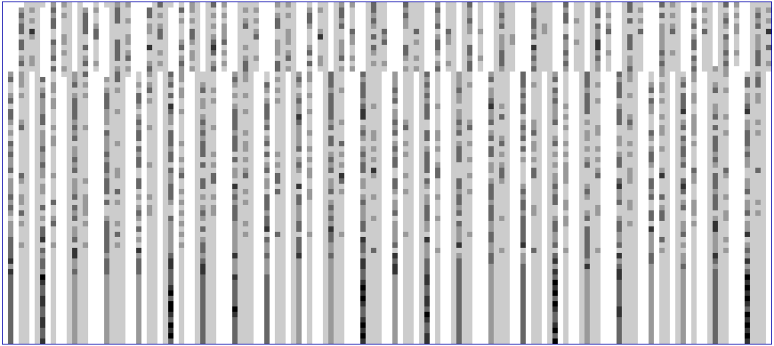
Introduce a new letter $0'$, and let $\tau(0') = 0$ and $\tau(n) = n$ for $n \in \mathbb{Z}_{\geq 0}$.

There exist words u, v of lengths $|u| = 1001 - 1$ and $|v| = 29949$ such that $\mathbf{w}_{6/5} = \tau(\varphi^\omega(0'))$, where

$$\varphi(n) = \begin{cases} v\,\varphi(0) & \text{if } n = 0' \\ u\,(n+2) & \text{if } n \geq 0. \end{cases}$$

$\mathbf{w}_{5/4}$ wrapped into 144 columns

$$\mathbf{w}_{5/4} = 000011110202101001011212000013110102101302\cdots$$



We don't know the structure of $\mathbf{w}_{5/4}$.

Catalogue

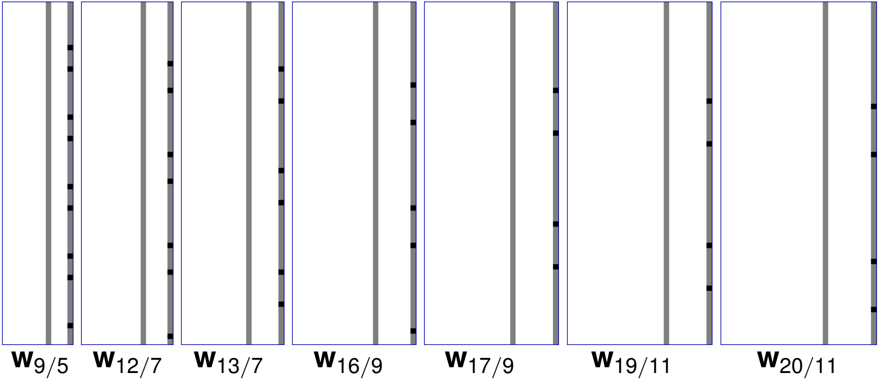
For many words $\mathbf{w}_{a/b}$, there is a related k-uniform morphism.

$$
\begin{aligned}
\tfrac{a}{b} = \tfrac{3}{2} \quad &\to \quad k = 6 \\
\tfrac{a}{b} = \tfrac{5}{3} \quad &\to \quad k = 7 \\
\tfrac{a}{b} = \tfrac{7}{4} \quad &\to \quad k = 50847 \\
\tfrac{a}{b} = \tfrac{8}{5} \quad &\to \quad k = 733 \\
\tfrac{a}{b} = \tfrac{6}{5} \quad &\to \quad k = 1001 \\
\tfrac{a}{b} = \tfrac{5}{4} \quad &\to \quad k = ?
\end{aligned}
$$

Question

Is there always a k-uniform morphism? How is k related to $\frac{a}{b}$?

A family related to $\mathbf{w}_{5/3}$



The interval $\frac{5}{3} \le \frac{a}{b} < 2$

Theorem

Let $\frac{5}{3} \le \frac{a}{b} < 2$ and b odd. Let φ be the $(2a-b)$-uniform morphism

$$\varphi(n) = 0^{a-1}\, 1\, 0^{a-b-1}\, (n+1)$$

for all $n \in \mathbb{Z}_{\ge 0}$. Then $\mathbf{w}_{a/b} = \varphi^{\omega}(0)$.

1. Show that φ preserves $\frac{a}{b}$-power-freeness.
   That is, if w is $\frac{a}{b}$-power-free then $\varphi(w)$ is $\frac{a}{b}$-power-free.
2. Show that φ preserves lex-leastness.
   That is, if decrementing any letter in w introduces an $\frac{a}{b}$-power, then decrementing any letter in $\varphi(w)$ introduces an $\frac{a}{b}$-power.

Since 0 is $\frac{a}{b}$-power-free and lex. least, it follows that $\mathbf{w}_{a/b} = \varphi^{\omega}(0)$.

Proving $\frac{a}{b}$-power-freeness

Definition

A k-uniform morphism φ locates words of length ℓ if there exists j such that, for all words $w, x \in \Sigma^*$ with $|x| = \ell$, every occurrence of the factor x in $\varphi(w)$ begins at a position congruent to j modulo k.

For example, $\varphi(n) = 000010(n+1)$ locates words of length 4.

Suppose φ locates words of length $|x|$.

If $\varphi(w)$ contains an $\frac{a}{b}$-power $(xy)^{a/b} = xyx$, then the two x's occur at positions that differ by $k \cdot m$.
By shifting, we conclude that w contains an $\frac{a}{b}$-power.

So if w is $\frac{a}{b}$-power-free, then $\varphi(w)$ does not contain long $\frac{a}{b}$-powers.

Other intervals

We have 30 symbolic $\frac{a}{b}$-power-free morphisms, found experimentally.

Theorem

Let $\frac{3}{2} < \frac{a}{b} < \frac{5}{3}$ *and* $\gcd(b, 5) = 1$*. The* $(5a - 4b)$*-uniform morphism*

$$\varphi(n) = 0^{a-1}\,1\,0^{a-b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,(n+1)$$

locates words of length $5a - 5b$ *and is* $\frac{a}{b}$*-power-free.*

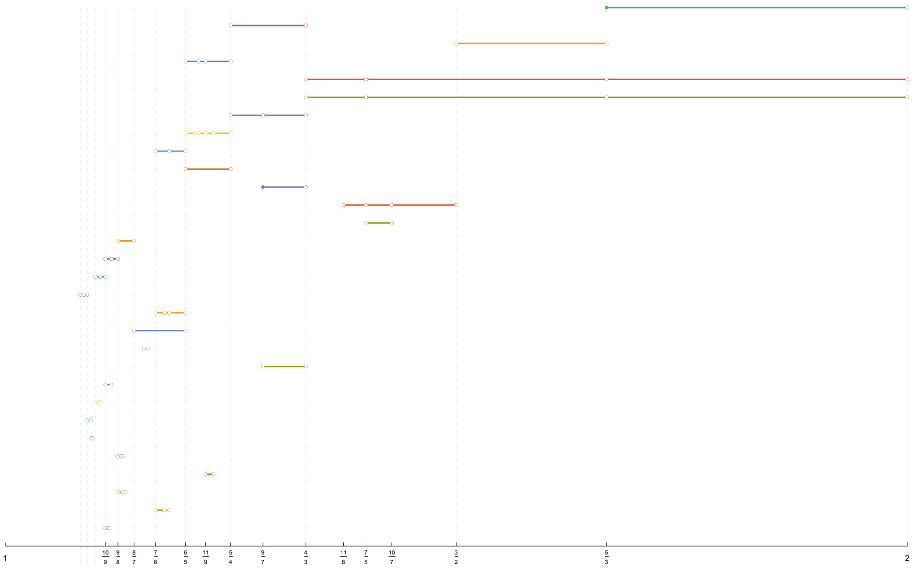
Theorem

Let $\frac{6}{5} < \frac{a}{b} < \frac{5}{4}$ *and* $\frac{a}{b} \notin \{\frac{11}{9}, \frac{17}{14}\}$*. The* a*-uniform morphism*

$$\varphi(n) = 0^{6a-7b-1}\,1\,0^{-3a+4b-1}\,1\,0^{-8a+10b-1}\,1\,0^{6a-7b-1}\,(n+1)$$

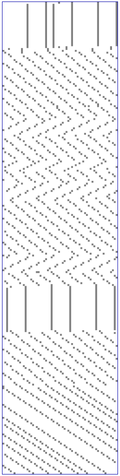
locates words of length a *and is* $\frac{a}{b}$*-power-free.*

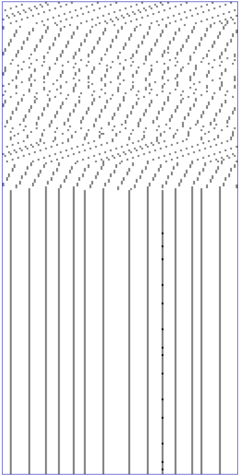
Coverage of $\frac{a}{b}$-power-free morphisms



$\mathbf{w}_{24/17}$ wrapped into 62 and 127 columns

Not all these morphisms are lexicographically least.
There exist rationals to which multiple theorems apply with different k.

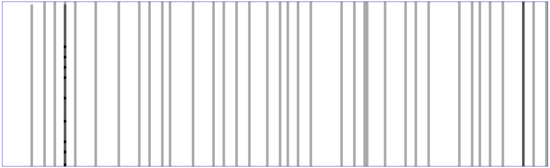




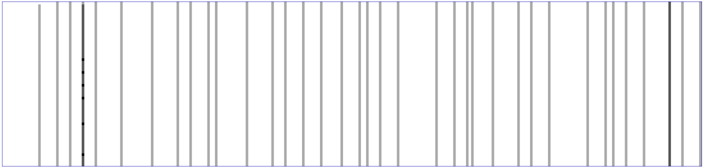
So $k = 127$ is some sort of attractor.

A family with a transient

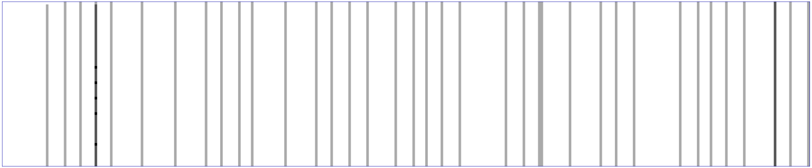
$\mathbf{w}_{17/13}$



$\mathbf{w}_{22/17}$



$\mathbf{w}_{25/19}$



Theorem

Let $\frac{9}{7} < \frac{a}{b} < \frac{4}{3}$ *and* $\gcd(b, 6) = 1$*. Let*

$$\varphi(0') = 0' 0^{a-2}\, 1\, 0^{a-b-1}\, 1\, 0^{a-b-1}\, 1 \varphi(0)$$

and

$$\begin{aligned}
\varphi(n) = {} & 0^{a-b-1}\, 1\, 0^{2a-2b-1}\, 1\, 0^{-a+2b-1}\, 1\, 0^{2a-2b-1}\, 1\, 0^{a-b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{4a-5b-1}\, 1 \\
& 0^{-a+2b-1}\, 1\, 0^{2a-2b-1}\, 1\, 0^{a-b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{5a-6b-1}\, 1 \\
& 0^{-2a+3b-1}\, 1\, 0^{4a-5b-1}\, 1\, 0^{a-b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{3a-3b-1}\, 1\, 0^{-2a+3b-1}\, 1 \\
& 0^{a-b-1}\, 1\, 0^{-3a+4b-1}\, 1\, 0^{5a-6b-1}\, 1\, 0^{2a-2b-1}\, 1\, 0^{a-b-1}\, 1\, 0^{-2a+3b-1}\, 1 \\
& 0^{3a-3b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{4a-5b-1}\, 1\, 0^{a-b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{2a-2b-1}\, 2 \\
& 0^{a-b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{3a-3b-1}\, 1\, 0^{-2a+3b-1}\, 1\, 0^{a-b-1}\, 1\, 0^{a-b-1}\, (n+2),
\end{aligned}$$

for $n \in \mathbb{Z}_{\geq 0}$*. Then* $\mathbf{w}_{a/b} = \tau(\varphi^\omega(0'))$*.*

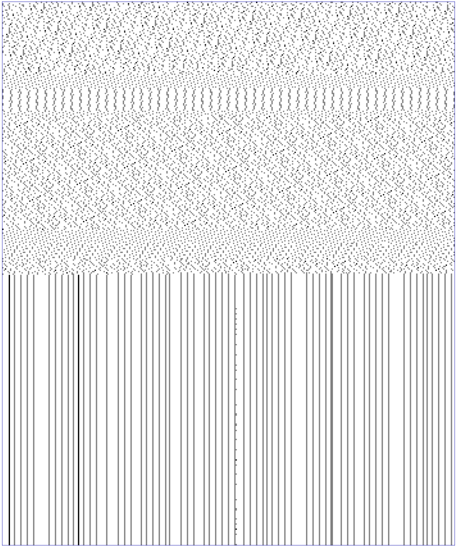
Sporadic rationals

The same proof technique applies to symbolic and explicit rationals...

The 50847-uniform morphism for $\mathbf{w}_{7/4}$ locates words of length 12940.
The 733-uniform morphism for $\mathbf{w}_{8/5}$ locates words of length 301.
The 45430-uniform morphism for $\mathbf{w}_{13/9}$ locates words of length 11400.
The 55657-uniform morphism for $\mathbf{w}_{17/10}$ locates words of length 37104.
etc.

Is there some way to make sense of them?

$\mathbf{w}_{27/23}$ wrapped into 353 columns



There exist words u, v on $\{0, 1, 2\}$ of lengths $|u| = 353 - 1$ and $|v| = 75019$ such that $\mathbf{w}_{27/23} = \tau(\varphi^\omega(0'))$, where

$$\varphi(n) = \begin{cases} v\,\varphi(0) & \text{if } n = 0' \\ u\,(n+0) & \text{if } n \geq 0. \end{cases}$$

$\mathbf{w}_{27/23}$ is also the lex least $\frac{27}{23}$-power-free word on $\{0, 1, 2\}$.

Workshop announcement

A U T O M A T I C S E Q U E N C E S . . .

Home page
Important dates

Invited speakers
Schedule
List of participants
Organizing committee

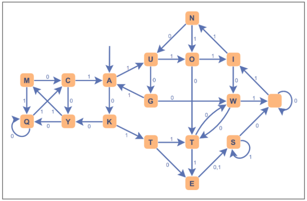
Registration
Accommodation
Useful information

Workshop on Automatic Sequences – Liège – May 2015

The University of Liège, Belgium, will host a workshop on automatic sequences the week of **May 25–29, 2015**. The theme of the workshop is automatic sequences and related areas, including:

- Automatic sequences in number theory
- Combinatorics on words
- Morphic sequences
- Regular sequences
- Symbolic dynamics
- Formal language theory

The program will include a mix of invited and contributed talks. If you are interested in giving a talk, email Eric Rowland at `erowland@ulg.ac.be`. In addition to talks, there will be some time for collaboration and discussion, so we will try to not pack the schedule too tightly.



Funding for the workshop comes from the University of Liège and the European Union.

Université de Liège

European Commission

`http://www.cant.ulg.ac.be/automatic2015`