Avoiding fractional powers over the natural numbers

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Squares and overlaps

A square is a nonempty word of the form ww.

An overlap is a word of the form wwc, where c is the first letter of w.

Squares are unavoidable on a binary alphabet:

010?

But Thue showed overlaps are avoidable:

$$\varphi^{\omega}(0) = 0110100110010110100101100110100 1\cdots$$

is overlap-free, where $\varphi(0) = 01$ and $\varphi(1) = 10$.

Lexicographically least words

What is the lex. least infinite word avoiding a pattern?

On a binary alphabet, the lex. least overlap-free word is

$$001001\varphi^{\omega}(1) = 001001100101100110100 1\cdots .$$

Open problem (Allouche–Shallit, *Automatic Sequences* §1.10)

Characterize the lex. least square-free word over $\{0, 1, 2\}$.

$$010201202101201020120210201021012010201202 10\cdots$$

Infinite alphabet

Guay-Paquet–Shallit (2009):

The lex. least square-free word on $\mathbb{Z}_{\geq 0}$ is

$$\mathbf{w}_2 = \varphi^\omega(0) = 010201030102010401020103010201 05 \cdots,$$

where φ is the 2-uniform morphism $\varphi(n) = 0(n+1)$.

The lex. least 5-power-free word on $\mathbb{Z}_{\geq 0}$ is

$$\mathbf{w}_5 = \varphi^\omega(0) = 000010000100001000010000200001 00001 \cdots,$$

where $\varphi(n) = 0000(n+1)$.

The lex. least overlap-free word on $\mathbb{Z}_{\geq 0}$ is also generated by a (non-uniform) morphism.

Fractional powers

Definition

A word w is an $\frac{a}{b}$-power if

$$w = v^e x$$

where $e \geq 0$ is an integer, x is a prefix of v, and $\frac{|w|}{|v|} = \frac{a}{b}$.

If $|x| = |y| = |z|$, then $xyzxyzx = (xyz)^{7/3}$ is a $\frac{7}{3}$-power.
$011101 = (0111)^{3/2}$ is a $\frac{3}{2}$-power.

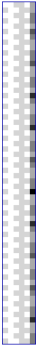
Notation

For $\frac{a}{b} > 1$, let $\mathbf{w}_{a/b}$ be the lex. least $\frac{a}{b}$-power-free word on $\mathbb{Z}_{\geq 0}$.

Avoiding $3/2$-powers

$$\mathbf{w}_{3/2} = 001102100112001103100113001102100114001103\cdots$$

$$\begin{aligned}\mathbf{w}_{3/2} = {} & 001102 \\ & 100112 \\ & 001103 \\ & 100113 \\ & 001102 \\ & 100114 \\ & 001103 \\ & 100112 \\ & \vdots\end{aligned}$$



Theorem (Rowland–Shallit 2012)

$\mathbf{w}_{3/2}$ *is a* 6*-regular sequence and generated by a* 6*-uniform morphism.*

Why 6?

The interval $\frac{a}{b} \geq 2$

$$\mathbf{w}_{5/2} = 000010000100001000010000200001000010000100 \cdots = \mathbf{w}_5$$

Let $\gcd(a, b) = 1$ from here on.

Theorem

If $\frac{a}{b} \geq 2$, then $\mathbf{w}_{a/b} = \mathbf{w}_a$.

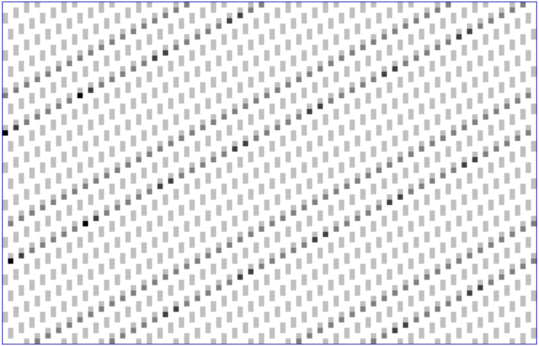
Proof (one direction).

The a-power $v^a = (v^b)^{a/b}$ is also an $\frac{a}{b}$-power.
So $\mathbf{w}_{a/b}$ is a-power-free. Thus $\mathbf{w}_a \leq \mathbf{w}_{a/b}$ lexicographically. $\square$

Therefore it suffices to consider $1 < \frac{a}{b} < 2$.

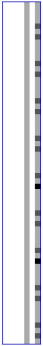
$\mathbf{w}_{5/3}$ wrapped into 100 columns

$$\mathbf{w}_{5/3} = 000010100001010000101000010100001020000101 \cdots$$



$\mathbf{w}_{5/3}$ wrapped into 7 columns

$$\mathbf{w}_{5/3} = 00001010000101000010100001010000102000010\cdots$$

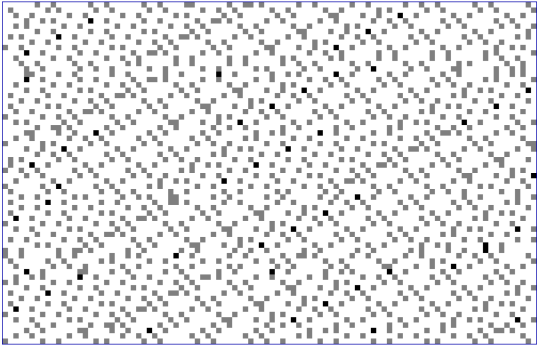


Theorem

$\mathbf{w}_{5/3} = \varphi^{\omega}(0)$, *where* $\varphi(n) = 000010(n+1)$ *is a* 7*-uniform morphism.*

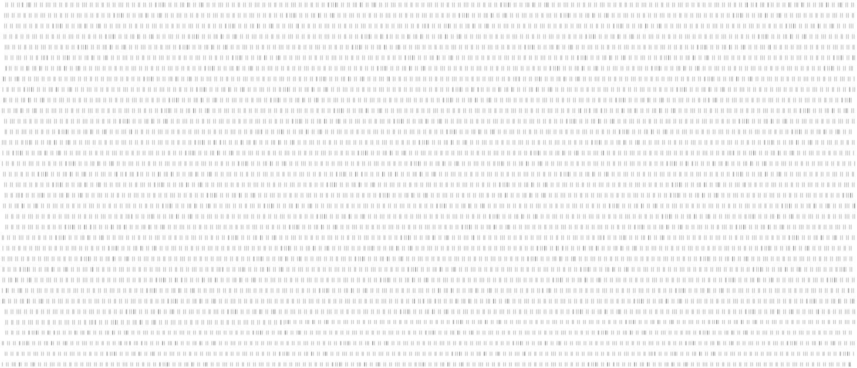
$\mathbf{w}_{7/4}$ wrapped into 100 columns

$$\mathbf{w}_{7/4} = 000000100100000010010000001001000011000000 \cdots$$



$\mathbf{w}_{7/4}$ wrapped into 50847 columns

$$\mathbf{w}_{7/4} = 000000100100000001001000000010010000011000000\cdots$$

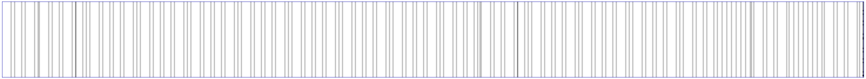


Theorem

$\mathbf{w}_{7/4} = \varphi^{\omega}(0)$ *for some* 50847*-uniform morphism* $\varphi(n) = u\,(n+2)$.

$\mathbf{w}_{8/5}$ wrapped into 733 columns

$$\mathbf{w}_{8/5} = 000000010010000001001000000001001100000000100\cdots$$

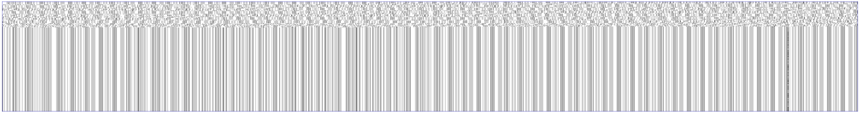


Theorem

$\mathbf{w}_{8/5} = \varphi^{\omega}(0)$ *for the* 733*-uniform morphism*

$$\begin{aligned}\varphi(n) = {} & 0000000100100000010010000000010011000000000100100000010010000000010020000 \\ & 0100100100000000100100000010010000001001000000001001001000000000100100000 \\ & 1001000000100100000000100100100000000100100000010010000001001000000001001 \\ & 0010000000010010000001001000000100100000000100100100000000100100000010010 \\ & 0000100100000000100100100000000100100000010010000001001000000001001001 00 \\ & 0000010010000001001000000100100000000100100100000000100100000010010000 01 \\ & 0010110000000010010000001001000000001002000000100100100000000100100000010 \\ & 0100000010010000000010010010000000010010000001001000000100100000000100100 \\ & 1000000010010000001001000000100100000000100100100000000100100000010010 00 \\ & 0010010001000100010001000100010001101000000001001000000100100000000101 \\ & 0001000100010001000100010001010000000010010000001001000000000010100(n+2).\end{aligned}$$

$$\mathbf{w}_{6/5} = 000001111102020201011101000202120210110010\cdots$$



There is a transient region.

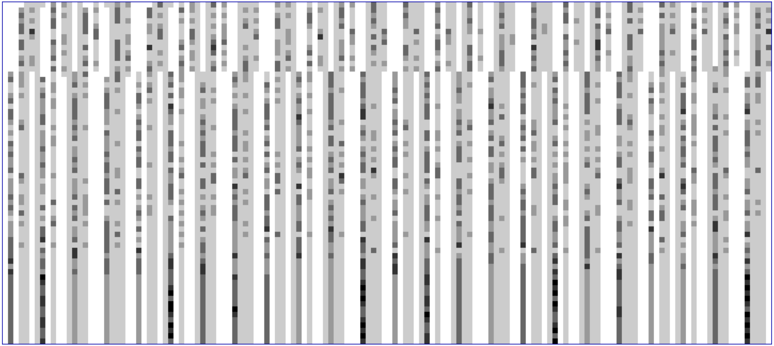
Introduce a new letter $0'$, and let $\tau(0') = 0$ and $\tau(n) = n$ for $n \in \mathbb{Z}_{\geq 0}$.

There exist words u, v of lengths $|u| = 1001 - 1$ and $|v| = 29949$ such that $\mathbf{w}_{6/5} = \tau(\varphi^{\omega}(0'))$, where

$$\varphi(n) = \begin{cases} v\,\varphi(0) & \text{if } n = 0' \\ u\,(n+2) & \text{if } n \geq 0. \end{cases}$$

$\mathbf{w}_{5/4}$ wrapped into 144 columns

$$\mathbf{w}_{5/4} = 000011110202101001011212000013110102101302\cdots$$



We don't know the structure of $\mathbf{w}_{5/4}$.

Catalogue

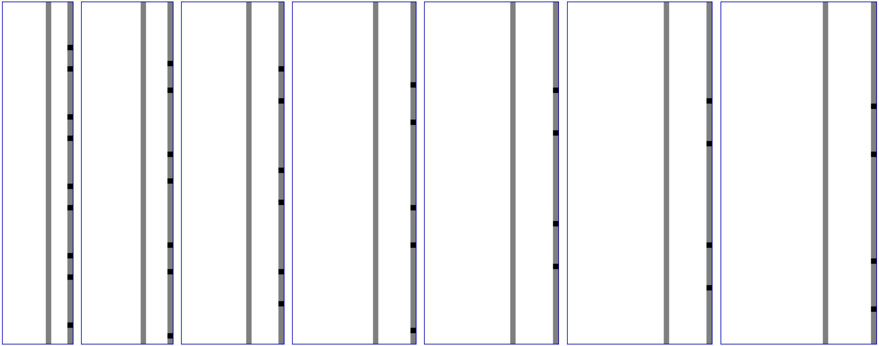
For many words $\mathbf{w}_{a/b}$, there is a related k-uniform morphism.

$$\begin{aligned}
\frac{a}{b} = \frac{3}{2} \quad &\to \quad k = 6 \\
\frac{a}{b} = \frac{5}{3} \quad &\to \quad k = 7 \\
\frac{a}{b} = \frac{7}{4} \quad &\to \quad k = 50847 \\
\frac{a}{b} = \frac{8}{5} \quad &\to \quad k = 733 \\
\frac{a}{b} = \frac{6}{5} \quad &\to \quad k = 1001 \\
\frac{a}{b} = \frac{5}{4} \quad &\to \quad k = ?
\end{aligned}$$

Question

Is there always a k-uniform morphism? How is k related to $\frac{a}{b}$?

A family related to $\mathbf{w}_{5/3}$



$\mathbf{w}_{9/5}$ $\mathbf{w}_{12/7}$ $\mathbf{w}_{13/7}$ $\mathbf{w}_{16/9}$ $\mathbf{w}_{17/9}$ $\mathbf{w}_{19/11}$ $\mathbf{w}_{20/11}$

The interval $\frac{5}{3} \le \frac{a}{b} < 2$

Theorem

Let $\frac{5}{3} \le \frac{a}{b} < 2$ and b odd. Let φ be the $(2a - b)$-uniform morphism

$$\varphi(n) = 0^{a-1}\, 1\, 0^{a-b-1}\, (n+1)$$

for all $n \in \mathbb{Z}_{\ge 0}$. Then $\mathbf{w}_{a/b} = \varphi^\omega(0)$.

1. Show that φ preserves $\frac{a}{b}$-power-freeness.
   That is, if w is $\frac{a}{b}$-power-free then $\varphi(w)$ is $\frac{a}{b}$-power-free.
2. Show that φ preserves lex-leastness.
   That is, if decrementing any letter in w introduces an $\frac{a}{b}$-power, then decrementing any letter in $\varphi(w)$ introduces an $\frac{a}{b}$-power.

Since 0 is $\frac{a}{b}$-power-free and lex. least, it follows that $\mathbf{w}_{a/b} = \varphi^\omega(0)$.

Proving $\frac{a}{b}$-power-freeness

Definition

A k-uniform morphism φ locates words of length ℓ if there exists j such that, for all words $w, x \in \Sigma^*$ with $|x| = \ell$, every occurrence of the factor x in $\varphi(w)$ begins at a position congruent to j modulo k.

For example, $\varphi(n) = 000010(n+1)$ locates words of length 4.

Suppose φ locates words of length $|x|$.

If $\varphi(w)$ contains an $\frac{a}{b}$-power $(xy)^{a/b} = xyx$, then the two x's occur at positions that differ by $k \cdot m$.
By shifting, we conclude that w contains an $\frac{a}{b}$-power.

So if w is $\frac{a}{b}$-power-free, then $\varphi(w)$ does not contain long $\frac{a}{b}$-powers.

Other intervals

We have 30 symbolic $\frac{a}{b}$-power-free morphisms, found experimentally.

Theorem

Let $\frac{3}{2} < \frac{a}{b} < \frac{5}{3}$ *and* $\gcd(b, 5) = 1$*. The* $(5a - 4b)$*-uniform morphism*

$$\varphi(n) = 0^{a-1}\,1\,0^{a-b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,(n+1)$$

locates words of length $5a - 5b$ *and is* $\frac{a}{b}$*-power-free.*

Theorem

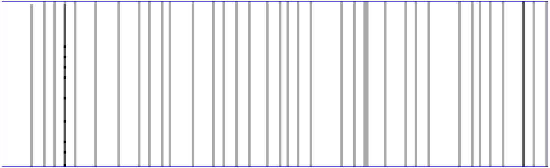
Let $\frac{6}{5} < \frac{a}{b} < \frac{5}{4}$ *and* $\frac{a}{b} \notin \{\frac{11}{9}, \frac{17}{14}\}$*. The* a*-uniform morphism*

$$\varphi(n) = 0^{6a-7b-1}\,1\,0^{-3a+4b-1}\,1\,0^{-8a+10b-1}\,1\,0^{6a-7b-1}\,(n+1)$$

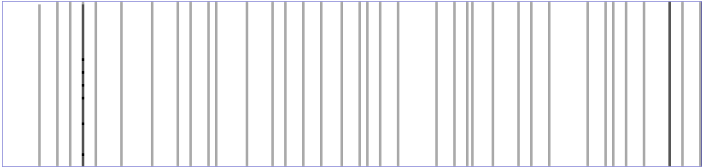
locates words of length a *and is* $\frac{a}{b}$*-power-free.*

A family with a transient

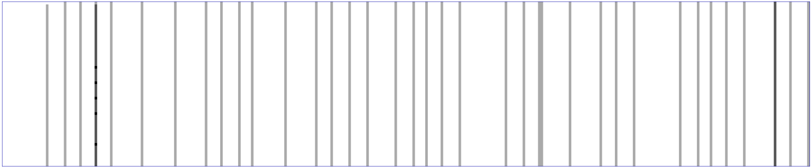
$\mathbf{w}_{17/13}$



$\mathbf{w}_{22/17}$



$\mathbf{w}_{25/19}$



Theorem

Let $\frac{9}{7} < \frac{a}{b} < \frac{4}{3}$ *and* $\gcd(b, 6) = 1$. *Let*

$$\varphi(0') = 0' 0^{a-2} \, 1 \, 0^{a-b-1} \, 1 \, 0^{a-b-1} \, 1 \varphi(0)$$

and

$$\begin{aligned}
\varphi(n) = {} & 0^{a-b-1}\,1\,0^{2a-2b-1}\,1\,0^{-a+2b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{4a-5b-1}\,1 \\
& 0^{-a+2b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{-2a+3b-1}\,1\,0^{5a-6b-1}\,1 \\
& 0^{-2a+3b-1}\,1\,0^{4a-5b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{3a-3b-1}\,1\,0^{-2a+3b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-3a+4b-1}\,1\,0^{5a-6b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1 \\
& 0^{3a-3b-1}\,1\,0^{-2a+3b-1}\,1\,0^{4a-5b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{2a-2b-1}\,2 \\
& 0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{3a-3b-1}\,1\,0^{-2a+3b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,(n+2),
\end{aligned}$$

for $n \in \mathbb{Z}_{\geq 0}$. *Then* $\mathbf{w}_{a/b} = \tau(\varphi^{\omega}(0'))$.

Sporadic rationals

The same proof technique applies to symbolic and explicit rationals...

The 50847-uniform morphism for $\mathbf{w}_{7/4}$ locates words of length 12940.
The 733-uniform morphism for $\mathbf{w}_{8/5}$ locates words of length 301.
The 45430-uniform morphism for $\mathbf{w}_{13/9}$ locates words of length 11400.
The 55657-uniform morphism for $\mathbf{w}_{17/10}$ locates words of length 37104.
etc.

Is there some way to make sense of them?

Credits

Thanks for your attention!

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