Avoiding fractional powers on an infinite alphabet

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Avoiding squares

Axel Thue (1863–1922)

A square is a nonempty word of the form xx. For example: $00, 0101$.
Are there arbitrarily long square-free words on $\{0, 1\}$?

Try to construct one:

010☒

Infinite alphabet

What is the lexicographically least square-free word on $\mathbb{Z}_{\geq 0}$?

$$0102010301020104010201030102010 5 \cdots$$

Theorem (Guay-Paquet–Shallit 2009)

Let $\varphi(n) = 0\,(n+1)$.
The lexicographically least square-free word on $\mathbb{Z}_{\geq 0}$ is $\varphi^\infty(0)$.

φ is 2-uniform.

$$\varphi(0) = 01$$
$$\varphi^2(0) = 0102$$
$$\varphi^3(0) = 01020103$$
$$\vdots$$

For each integer $a \geq 2$, let $\varphi(n) = 0^{a-1}(n+1)$.
The lexicographically least a-power-free word on $\mathbb{Z}_{\geq 0}$ is $\varphi^\infty(0)$.

Avoiding overlaps

An overlap is a word of the form xxc, where c is the first letter of x.
For example: $000, 01010$.

Overlaps are avoidable on a binary alphabet (Thue):

$$\varphi^{\infty}(0) = 01101001100101101001011001101001\cdots$$

is overlap-free, where $\varphi(0) = 01$ and $\varphi(1) = 10$.

$$\begin{aligned}\varphi(0) &= 01\\ \varphi^2(0) &= 0110\\ \varphi^3(0) &= 01101001\\ &\vdots\end{aligned}$$

Avoiding overlaps

Lexicographically least overlap-free word on $\mathbb{Z}_{\geq 0}$:

001	001	1001002	001	001	1001002	1001002	001	001	1001002	001	001	2001001100100200100110010003	$\cdots$
$\varphi(0)$	$\varphi(0)$	$\varphi(1)$	$\varphi(0)$	$\varphi(0)$	$\varphi(1)$	$\varphi(1)$	$\varphi(0)$	$\varphi(0)$	$\varphi(1)$	$\varphi(0)$	$\varphi(0)$	$\varphi(2)$	$\cdots$

Let σ be the right shift: $\sigma(xc) = cx$ for words x and letters c.

Theorem (Guay-Paquet–Shallit 2009)

Define φ recursively by $\varphi(n) = \sigma(\varphi^n(00))\,(n+1)$.
The lexicographically least overlap-free word on $\mathbb{Z}_{\geq 0}$ is $\varphi^\infty(0)$.

φ is non-uniform.

$$\varphi(0) = 001$$
$$\varphi^2(0) = 0010011001002$$
$$\vdots$$

Fractional powers

$01220 = (0122)^{5/4}$ is a $\frac{5}{4}$-power.
$011101 = (0111)^{3/2}$ is a $\frac{3}{2}$-power.

Definition

Let $\frac{a}{b} > 1$. A word w is an $\frac{a}{b}$-power if

$$w = (xy)^e x$$

and $\frac{|w|}{|xy|} = \frac{a}{b}$ for some words x, y and some integer $e \geq 1$.

$\frac{5}{4}$-powers look like xyx where $|y| = 3|x|$.
$\frac{3}{2}$-powers look like xyx where $|y| = |x|$.

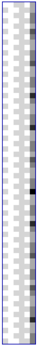
Notation

Let $\mathbf{w}_{a/b}$ be the lex. least $\frac{a}{b}$-power-free word on $\mathbb{Z}_{\geq 0}$.

We assume $\gcd(a, b) = 1$.

$$\mathbf{w}_{3/2} = 001102100112001103100113001102100114001103\cdots$$

$$\begin{aligned}\mathbf{w}_{3/2} = {} & 001102 \\ & 100112 \\ & 001103 \\ & 100113 \\ & 001102 \\ & 100114 \\ & 001103 \\ & 100112 \\ & \vdots\end{aligned}$$



Theorem (Rowland–Shallit 2012)

The ith letter $w(i)$ of $\mathbf{w}_{3/2}$ satisfies $w(6i+5) = w(i) + 2$.

Other notions of avoidance

Notation

- Let $\mathbf{w}_{\geq a/b}$ be the lex. least infinite word on $\mathbb{Z}_{\geq 0}$ avoiding $\frac{p}{q}$-powers for all $\frac{p}{q} \geq \frac{a}{b}$.
- Let $\mathbf{w}_{>a/b}$ be the lex. least infinite word on $\mathbb{Z}_{\geq 0}$ avoiding $\frac{p}{q}$-powers for all $\frac{p}{q} > \frac{a}{b}$.

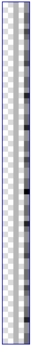
What are the relationships between $\mathbf{w}_{a/b}$, $\mathbf{w}_{\geq a/b}$, and $\mathbf{w}_{>a/b}$?

The lex. least overlap-free word is $\mathbf{w}_{>2}$.

Avoiding $\geq 3/2$-powers

$$\mathbf{w}_{\geq 3/2} = 012031021301204102140120310215012041021301203\cdots$$

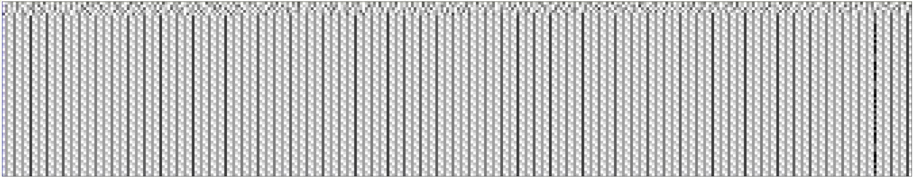
$$\begin{aligned} \mathbf{w}_{\geq 3/2} = {} & 01203 \\ & 10213 \\ & 01204 \\ & 10214 \\ & 01203 \\ & 10215 \\ & 01204 \\ & 10213 \\ & \vdots \end{aligned}$$



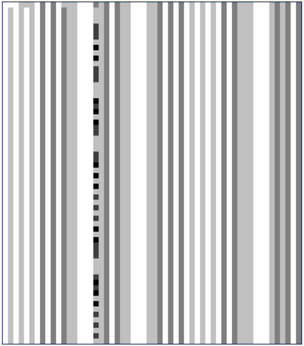
Theorem

We have $\mathbf{w}_{\geq 3/2}(5i+4) = \mathbf{w}_{3/2}(i) + 3$ *for all* $i \geq 0$.

$\mathbf{w}_{\geq 4/3}$:



$\mathbf{w}_{4/3}$:



Conjecture:
$\mathbf{w}_{\geq 4/3}(336i+1666) = \mathbf{w}_{4/3}(56i+17)+4$ for all $i \geq 0$.

Are there similar relationships between $\mathbf{w}_{\geq a/b}$ and $\mathbf{w}_{a/b}$ for other $\frac{a}{b}$?

We focus on $\mathbf{w}_{a/b}$.

The interval $\frac{a}{b} \geq 2$

$$\mathbf{w}_{5/2} = 000010000100001000010000200001000001\cdots = \mathbf{w}_5 = \varphi^\infty(0)$$

where $\varphi(n) = 0000(n+1)$.

Theorem

If $\frac{a}{b} \geq 2$, then $\mathbf{w}_{a/b} = \mathbf{w}_a$.

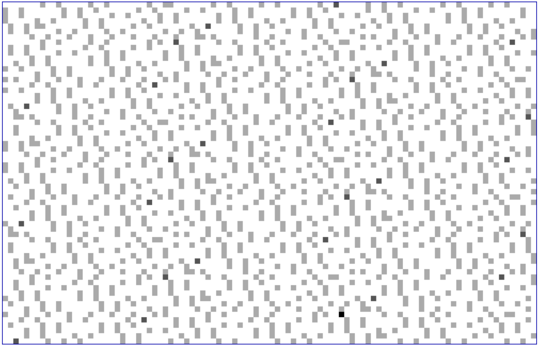
Proof (one direction).

The a-power $v^a = (v^b)^{a/b}$ is also an $\frac{a}{b}$-power.
So $\mathbf{w}_{a/b}$ is a-power-free. Thus $\mathbf{w}_a \leq \mathbf{w}_{a/b}$ lexicographically. □

Therefore it suffices to consider $1 < \frac{a}{b} < 2$.

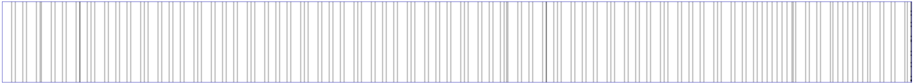
$\mathbf{w}_{8/5}$ wrapped into 100 columns

$$\mathbf{w}_{8/5} = 0000000100100000010010000000100110000000100\cdots$$



$\mathbf{w}_{8/5}$ wrapped into 733 columns

$$\mathbf{w}_{8/5} = 000000010010000001001000000001001100000000100\cdots$$



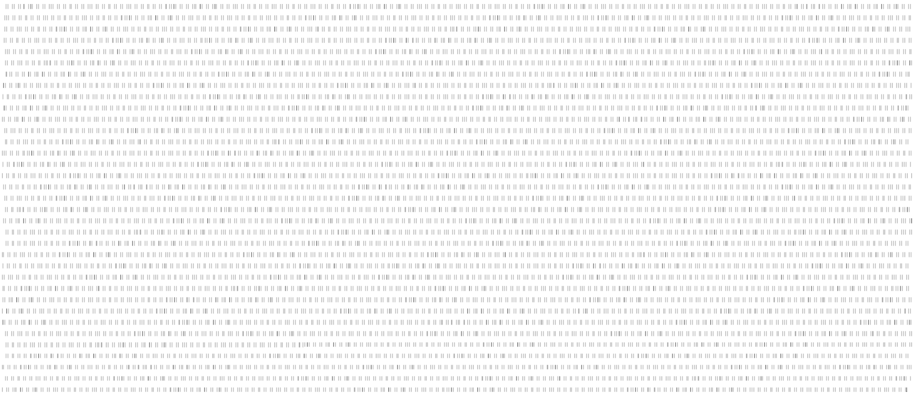
Theorem

$\mathbf{w}_{8/5} = \varphi^\infty(0)$ *for the* 733*-uniform morphism*

$$\begin{aligned}\varphi(n) = {} & 000000010010000001001000000001001100000000100100000010010000000010020000 \\ & 010010010000000010010000001001000000100100000000100100100000000100100000 \\ & 100100000010010000000010010010000000010010000001001000000100100000001001 \\ & 001000000001001000000100100000010010000000010010010000000010010000010010 \\ & 000010010000000010010010000000010010000001001000000100100000000100100100 \\ & 000001001000000100100000010010000000010010010000000010010000001001000001 \\ & 001011000000001001000000100100000000100200000010010010000000010010000010 \\ & 010000001001000000001001001000000001001000000100100000010010000000100100 \\ & 100000001001000000100100000010010000000010010010000000010010000001001000 \\ & 001001000100010001000100010001000110100000000100100000010010000000101 \\ & 0001000100010001000100010001010000000010010000001001000000010100(n+2).\end{aligned}$$

$\mathbf{w}_{7/4}$ wrapped into 50847 columns

$$\mathbf{w}_{7/4} = 000000100100000001001000000010010000011000000 \cdots$$

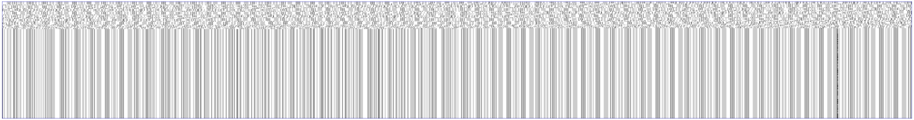


Theorem

$\mathbf{w}_{7/4} = \varphi^{\infty}(0)$ *for some* 50847*-uniform morphism* $\varphi(n) = u\,(n+2)$.

$\mathbf{w}_{6/5}$ wrapped into 1001 columns

$$\mathbf{w}_{6/5} = 000001111102020201011101000202120210110010\cdots$$



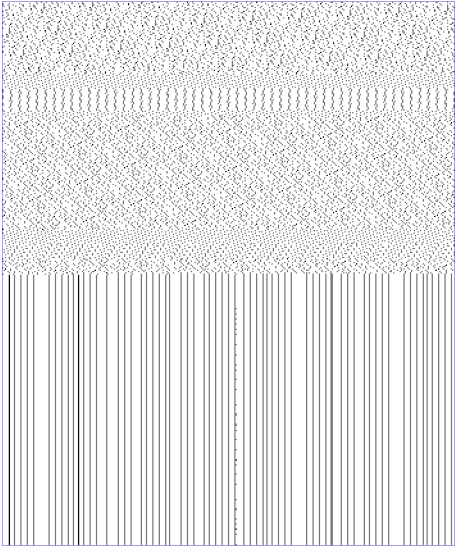
There is a transient region.

Introduce a new letter $0'$, and let $\tau(0') = 0$ and $\tau(n) = n$ for $n \in \mathbb{Z}_{\geq 0}$.

There exist words u, v of lengths $|u| = 1000$ and $|v| = 29949$ such that $\mathbf{w}_{6/5} = \tau(\varphi^{\infty}(0'))$, where

$$\varphi(n) = \begin{cases} v\,\varphi(0) & \text{if } n = 0' \\ u\,(n+2) & \text{if } n \in \mathbb{Z}. \end{cases}$$

$\mathbf{w}_{27/23}$ wrapped into 353 columns



There exist words u, v on $\{0, 1, 2\}$ of lengths $|u| = 352$ and $|v| = 75019$ such that $\mathbf{w}_{27/23} = \tau(\varphi^\omega(0'))$, where

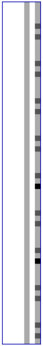
$$\varphi(n) = \begin{cases} v\,\varphi(0) & \text{if } n = 0' \\ u\,(n + 0) & \text{if } n \in \mathbb{Z}. \end{cases}$$

$\mathbf{w}_{27/23}$ is also the lex. least $\frac{27}{23}$-power-free word on $\{0, 1, 2\}$.

$\mathbf{w}_{5/3}$

$$\mathbf{w}_{5/3} = 000010100001010000101000010100001020000101\cdots$$

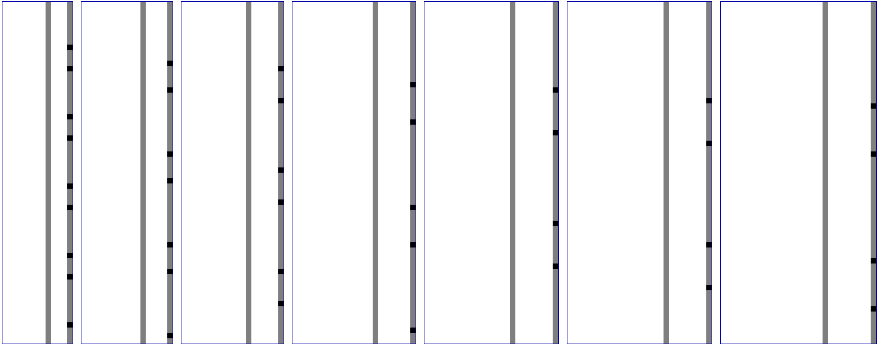
$$\begin{aligned}\mathbf{w}_{5/3} = {} & 0000101 \\ & 0000101 \\ & 0000101 \\ & 0000101 \\ & 0000102 \\ & 0000101 \\ & 0000102 \\ & 0000101 \\ & \vdots\end{aligned}$$



$$w(7i+6) = w(i) + 1$$

$\mathbf{w}_{5/3} = \varphi^{\infty}(0)$, where $\varphi(n) = 000010(n+1)$ is a 7-uniform morphism.

A family related to $\mathbf{w}_{5/3}$



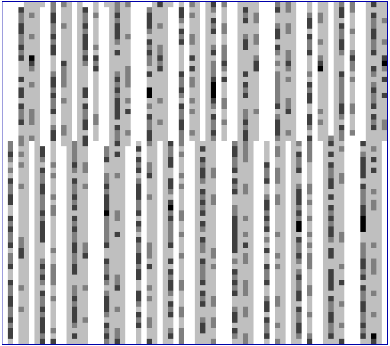
$\mathbf{w}_{9/5}$ $\mathbf{w}_{12/7}$ $\mathbf{w}_{13/7}$ $\mathbf{w}_{16/9}$ $\mathbf{w}_{17/9}$ $\mathbf{w}_{19/11}$ $\mathbf{w}_{20/11}$

Theorem (Pudwell–Rowland 2018)

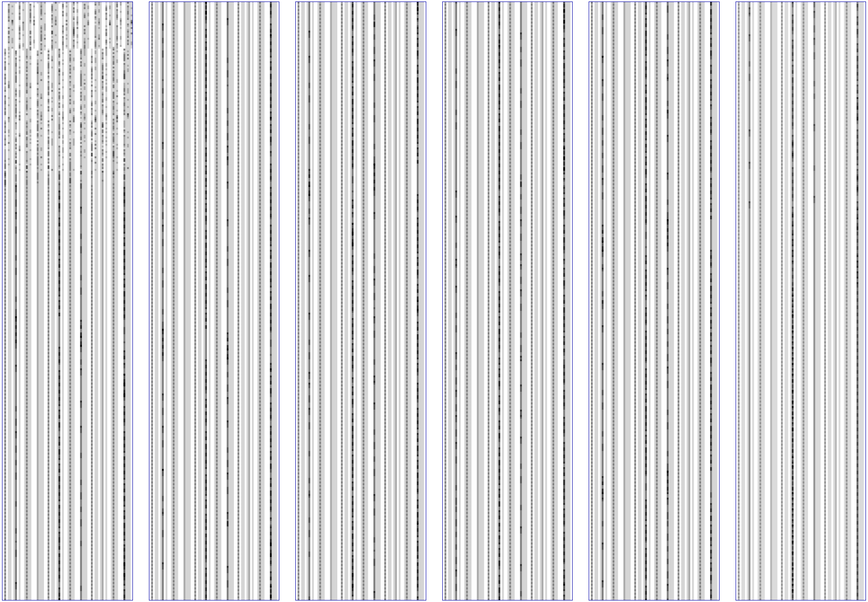
Let $\frac{5}{3} \leq \frac{a}{b} < 2$ with b odd. Then $\mathbf{w}_{a/b} = \varphi^\infty(0)$, where $\varphi(n) = 0^{a-1}\,1\,0^{a-b-1}\,(n+1)$ is a $(2a-b)$-uniform morphism.

$\mathbf{w}_{5/4}$ wrapped into 72 columns

$$\mathbf{w}_{5/4} = 00001111020210100101121200001311010210130 2\cdots$$



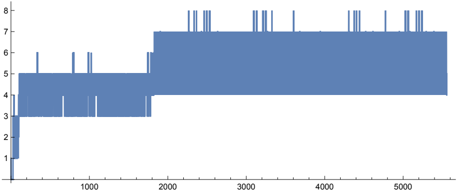
$\mathbf{w}_{5/4}$ — first 2000 rows



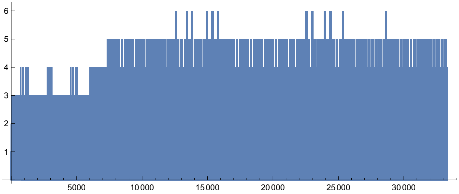
Guessing a recurrence

Let $w(i)$ be the ith letter of $\mathbf{w}_{5/4}$.

$w(72i+31)_{i\geq 0}$:



$w(i)_{i\geq 0}$:

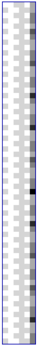


Implied relationship:

$$w(6i+123061) = w(i+5920) + \begin{cases} 3 & \text{if } i \equiv 0,2 \mod 8 \\ 1 & \text{if } i \equiv 4,6 \mod 8 \\ 2 & \text{if } i \equiv 1 \mod 2 \end{cases}$$

Morphism for $\mathbf{w}_{3/2}$

$\mathbf{w}_{3/2} = 001102100112001103100113001102100114001103\cdots$



Use two kinds of letters.
Alphabet: $\Sigma_2 = \{n_j : n \in \mathbb{Z}, j \in \{0,1\}\}$
Coding: $\tau(n_j) = n$

6-uniform morphism:
$\varphi(n_0) = 0_0 0_1 1_0 1_1 0_0 (n+2)_1$
$\varphi(n_1) = 1_0 0_1 0_0 1_1 1_0 (n+2)_1$

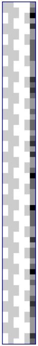
Morphic description: $\mathbf{w}_{3/2} = \tau(\varphi^\infty(0_0))$.

Morphism for $\mathbf{w}_{5/4}$

Theorem (Rowland–Stipulanti 2020)

There exist words p, z of lengths $|p| = 6764$ and $|z| = 20226$ such that $\mathbf{w}_{5/4} = p\,\tau(\varphi(z)\varphi^2(z)\cdots)$.

z is a word on $\Sigma_8 = \{n_j : n \in \mathbb{Z}, 0 \leq j \leq 7\}$. z contains -1_0 and -1_2.



$$\begin{aligned}
\varphi(n_0) &= 0_0 1_1 0_2 0_3 1_4 (n+3)_5 \\
\varphi(n_1) &= 1_6 1_7 0_0 0_1 0_2 (n+2)_3 \\
\varphi(n_2) &= 1_4 1_5 1_6 0_7 0_0 (n+3)_1 \\
\varphi(n_3) &= 0_2 1_3 1_4 0_5 1_6 (n+2)_7 \\
\varphi(n_4) &= 0_0 1_1 0_2 0_3 1_4 (n+1)_5 \\
\varphi(n_5) &= 1_6 1_7 0_0 0_1 0_2 (n+2)_3 \\
\varphi(n_6) &= 1_4 1_5 1_6 0_7 0_0 (n+1)_1 \\
\varphi(n_7) &= 0_2 1_3 1_4 0_5 1_6 (n+2)_7
\end{aligned}$$

Proof outline

1. Show that $p\,\tau(\varphi(z)\varphi^2(z)\cdots)$ avoids $\frac{5}{4}$-powers.
2. Show that decreasing any letter in $p\,\tau(\varphi(z)\varphi^2(z)\cdots)$ introduces a $\frac{5}{4}$-power ending at that position.

For previously studied words $\mathbf{w}_{a/b}$, Step 1 involves showing that φ is $\frac{a}{b}$-power-free. That is, if w is $\frac{a}{b}$-power-free then $\varphi(w)$ is $\frac{a}{b}$-power-free.

However, the morphism for $\mathbf{w}_{5/4}$ is not $\frac{5}{4}$-power-free:
For $n, m \in \mathbb{Z}$, the word $0_4 n_5 m_6$ is $\frac{5}{4}$-power-free, but its image is not:

$$\varphi(0_4 n_5 m_6) = 0_0 1_1 0_2 0_3 1_4 1_5 \;\; 1_6 1_7 0_0 0_1 0_2 (n+2)_3 \;\; 1_4 1_5 1_6 0_7 0_0 (m+1)_1$$

Pre-$\frac{5}{4}$-powers

A word w is a pre-$\frac{5}{4}$-power if $\varphi(w)$ is a $\frac{5}{4}$-power.
For example, $0_0 n_1 n_2 n_3 2_4$ is a pre-$\frac{5}{4}$-power:

$$\varphi(0_0 n_1 n_2 n_3 2_4) = 0_0 1_1 0_2 0_3 1_4 (0+3)_5 \;\; \varphi(n_1 n_2 n_3) \;\; 0_0 1_1 0_2 0_3 1_4 (2+1)_5.$$

Every $\frac{5}{4}$-power is a pre-$\frac{5}{4}$-power.

Idea: Show that φ preserves pre-$\frac{5}{4}$-power-freeness.

Let Γ be the set

$$\{-3_0, -3_2, -2_0, -2_1, -2_2, -2_3, -2_5, -2_7, -1_1, -1_3, -1_4, -1_5, -1_6, -1_7, 0_4, 0_6\}.$$

Proposition

If w is a pre-$\frac{5}{4}$-power-free subscript-increasing word on $\Sigma_8 \setminus \Gamma$, then $\varphi(w)$ is pre-$\frac{5}{4}$-power-free.

z is a subscript-increasing word on $\Sigma_8 \setminus \Gamma$.

Proof strategy

1. Sequence of results establishing $\frac{5}{4}$-power-freeness:

   Theorem. $z\varphi(z)\varphi^2(z)\cdots$ is pre-$\frac{5}{4}$-power-free.

   Corollary. $\varphi(z)\varphi^2(z)\varphi^3(z)\cdots$ is pre-$\frac{5}{4}$-power-free.

   Corollary. $\varphi(z)\varphi^2(z)\varphi^3(z)\cdots$ is $\frac{5}{4}$-power-free.

   Lemma. $\tau(\varphi(z)\varphi^2(z)\varphi^3(z)\cdots)$ is $\frac{5}{4}$-power-free.

   Theorem. $p\tau(\varphi(z)\varphi^2(z)\varphi^3(z)\cdots)$ is $\frac{5}{4}$-power-free.

2. For establishing lexicographic leastness:

   Case analysis and complicated induction.

Both steps involve large finite checks carried out by computer.

k-regular sequences

A sequence $w(i)_{i\geq 0}$ of rational numbers is k-regular if the set

$$\left\{w(k^e i + r)_{i\geq 0} : e \geq 0 \text{ and } 0 \leq r \leq k^e - 1\right\}$$

is contained in a finite-dimensional $\mathbb{Q}$-vector space of sequences.

Example

Let $w(i)$ be the ith letter of $\mathbf{w}_2 = 0102010301020104\cdots$.
Then $w(2i) = 0$ and $w(2i+1) = w(i) + 1$. It follows that

$$\begin{aligned}
w(4i+0) &= 0 \\
w(4i+1) &= w(2(2i)+1) = w(2i) + 1 = 1 \\
w(4i+2) &= 0 \\
w(4i+3) &= w(2(2i+1)+1) = w(2i+1) + 1 = w(i) + 2.
\end{aligned}$$

The sequences $w(i)_{i\geq 0}$ and $(1)_{i\geq 0}$ generate a $\mathbb{Q}$-vector space containing each $w(k^e i + r)_{i\geq 0}$. Therefore $w(i)_{i\geq 0}$ is 2-regular.

Regularity from a recurrence

Theorem

Let $k \geq 2$ and $\ell \geq 1$.
Let $d(i)_{i \geq 0}$ and $u(i)_{i \geq 0}$ be periodic integer sequences with period lengths ℓ and $k\ell$, respectively.
Let r, s be nonnegative integers such that $r - s + k - 1 \geq 0$.
Let $w(i)_{i \geq 0}$ be an integer sequence such that, for all $0 \leq m \leq k - 1$ and all $i \geq 0$,

$$w(ki + r + m) = \begin{cases} u(ki + m) & \text{if } 0 \leq m \leq k - 2 \\ w(i + s) + d(i) & \text{if } m = k - 1. \end{cases}$$

Then $w(i)_{i \geq 0}$ is k-regular.

Theorem

The sequence of letters in $\mathbf{w}_{5/4}$ is a 6-regular sequence with rank 188.

Catalog of $\mathbf{w}_{a/b}$

General recurrence for self-similar column: $w(ki+r') = w(i+s) + d(i)$.

a/b	k	$d(i)$	r'	s	rank	note
$a \in \mathbb{Z}_{\geq 2}$	a	1	0	0	2	
$3/2$	6	2	0	0	3	
$4/3$	56	$1, 2$	73	0	4	
$5/3$	7	1	0	0	2	
$5/4$	6	$1, 2, 3$	123061	5920	188	
$7/4$	50847	2	0	0	2	
$6/5$	1001	3	30949	0	33	
$7/5$	80874	1	173978	0		conjectural
$8/5$	733	2	0	0	2	
$9/5$	13	1	0	0	2	
$7/6$	41190	3	41201	0		conjectural
$11/6$						[no conjecture]

Is every word $\mathbf{w}_{a/b}$ k-regular for some k?

Morphisms

Morphism for $\mathbf{w}_{3/2}$:

$$\begin{aligned}\varphi(n_0) &= 0_0 0_1 1_0 1_1 0_0 (n+2)_1\\ \varphi(n_1) &= 1_0 0_1 0_0 1_1 1_0 (n+2)_1\end{aligned}$$

Morphism for $\mathbf{w}_{5/4}$:

$$\begin{aligned}\varphi(n_0) &= 0_0 1_1 0_2 0_3 1_4 (n+3)_5\\ \varphi(n_1) &= 1_6 1_7 0_0 0_1 0_2 (n+2)_3\\ \varphi(n_2) &= 1_4 1_5 1_6 0_7 0_0 (n+3)_1\\ \varphi(n_3) &= 0_2 1_3 1_4 0_5 1_6 (n+2)_7\\ \varphi(n_4) &= 0_0 1_1 0_2 0_3 1_4 (n+1)_5\\ \varphi(n_5) &= 1_6 1_7 0_0 0_1 0_2 (n+2)_3\\ \varphi(n_6) &= 1_4 1_5 1_6 0_7 0_0 (n+1)_1\\ \varphi(n_7) &= 0_2 1_3 1_4 0_5 1_6 (n+2)_7\end{aligned}$$

Which are more natural — the morphisms or the lex. least words?

References

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- Lara Pudwell and Eric Rowland, Avoiding fractional powers over the natural numbers, *The Electronic Journal of Combinatorics* **25** (2018) #P2.27.
- Eric Rowland and Jeffrey Shallit, Avoiding $3/2$-powers over the natural numbers, *Discrete Mathematics* **312** (2012) 1282–1288.
- Eric Rowland and Manon Stipulanti, Avoiding $5/4$-powers on the alphabet of non-negative integers, *The Electronic Journal of Combinatorics* **27** (2020) #P3.42.