Automatic proofs for establishing the structure of integer sequences avoiding a pattern

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Squares on a 3-letter alphabet

A square is a nonempty word of the form $w^2 = ww$.
Are squares are avoidable on a 3-letter alphabet?

Axel Thue (1863–1922)

Are there arbitrarily long square-free words on $\{0, 1, 2\}$?

Choose an order on $\{0, 1, 2\}$ and try to construct one:

$$01020120210120102012021020102101201020120210 \cdots$$

The backtracking algorithm builds the lexicographically least sequence (if it exists).

Squares on an infinite alphabet

On an infinite alphabet, the backtracking algorithm doesn't backtrack.

Are squares avoidable on $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$? Yes.

$$\mathbf{s}_2 = 0102010301020104010201030102010 5\cdots$$

Let $\varphi(n) = 0\,(n+1)$ for each $n \in \mathbb{Z}_{\geq 0}$.

$$\varphi(0) = 01$$
$$\varphi^2(0) = 0102$$
$$\varphi^3(0) = 01020103$$
$$\vdots$$
$$\varphi^\infty(0) = 0102010301020104010201030102010 5\cdots$$

Since $|\varphi(n)| = 2$, we say φ is a 2-uniform morphism.

Fractional powers

$01110111 = (0111)^2$ is a square.
$011101 = (0111)^{3/2}$ is a $\frac{3}{2}$-power.

$\texttt{abracadabra} = (\texttt{abracad})^{11/7}$ is an $\frac{11}{7}$-power.

Definition

A word w is an $\frac{a}{b}$-power if

$$w = v^e x$$

where $e \geq 0$ is an integer, x is a prefix of v, and $\frac{|w|}{|v|} = \frac{a}{b}$.

Notation

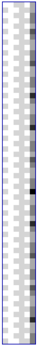
For $\frac{a}{b} > 1$, let $\mathbf{s}_{a/b}$ be the lex. least $\frac{a}{b}$-power-free sequence on $\mathbb{Z}_{\geq 0}$.

We assume $\gcd(a, b) = 1$ from now on.

Avoiding 3/2-powers

$\mathbf{s}_{3/2} = 001102100112001103100113001102100114001103\cdots$

$$\mathbf{s}_{3/2} = \begin{array}{l} 001102 \\ 100112 \\ 001103 \\ 100113 \\ 001102 \\ 100114 \\ 001103 \\ 100112 \\ \vdots \end{array}$$



$$s(6n+5) = s(n) + 2$$

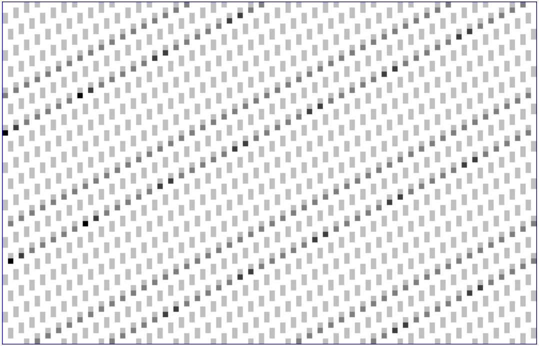
Theorem (Rowland–Shallit 2012)

The sequence $\mathbf{s}_{3/2}$ is generated by a 6-uniform morphism.

Why 6?

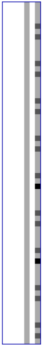
$\mathbf{s}_{5/3}$ wrapped into 100 columns

$$\mathbf{s}_{5/3} = 000010100001010000101000010100001020000101\cdots$$



$\mathbf{s}_{5/3}$ wrapped into 7 columns

$$\mathbf{s}_{5/3} = 0000101000010100001010000101000010200001 01 \cdots$$

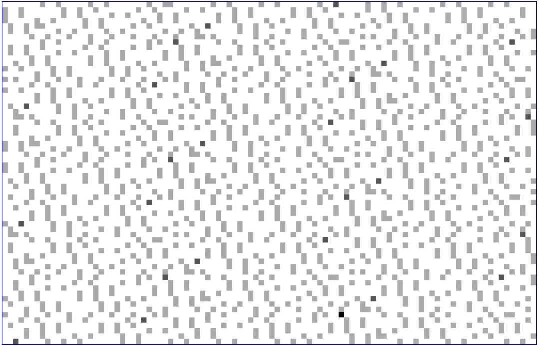


Theorem

$\mathbf{s}_{5/3} = \varphi^\infty(0)$, *where* $\varphi(n) = 000010(n+1)$ *is a 7-uniform morphism.*

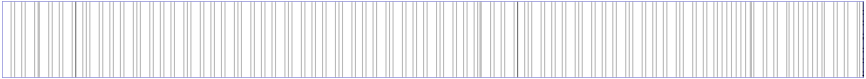
$\mathbf{s}_{8/5}$ wrapped into 100 columns

$\mathbf{s}_{8/5} = 00000001001000000100100000001001100000000100\cdots$



$\mathbf{s}_{8/5}$ wrapped into 733 columns

$$\mathbf{s}_{8/5} = 00000001001000000100100000000100110000000100\cdots$$



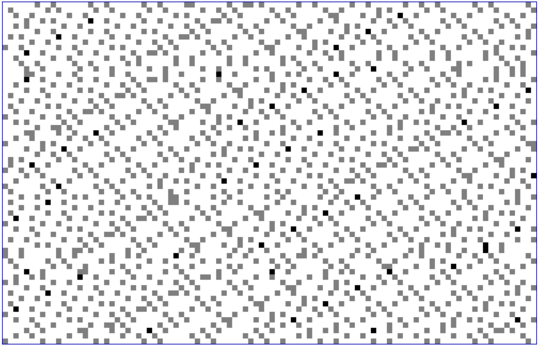
Theorem

$\mathbf{s}_{8/5} = \varphi^\infty(0)$ *for the 733-uniform morphism*

$$\begin{aligned}\varphi(n) = {} & 00000001001000000100100000000100110000000100100000010010000000010020000 \\ & 01001001000000001001000000100100000010010000000010010010000000100100000 \\ & 10010000001001000000001001001000000001001000000100100000010010000000001001 \\ & 00100000000100100000010010000001001000000001001001000000001001000000010010 \\ & 00001001000000001001001000000001001000000100100000010010000000100100100 \\ & 00000100100000010010000001001000000001001001000000001001000000100100001 \\ & 00101100000000100100000010010000000010020000001001001000000001001000000010 \\ & 01000000100100000000100100100000000100100000010010000001001000000000100100 \\ & 10000000100100000010010000001001000000001001001000000001001000000100100 0 \\ & 00100100010001000100010001000100011010000000010010000001001000000000101 \\ & 00010001000100010001000100010100000000100100000010010000000010100(n+2).\end{aligned}$$

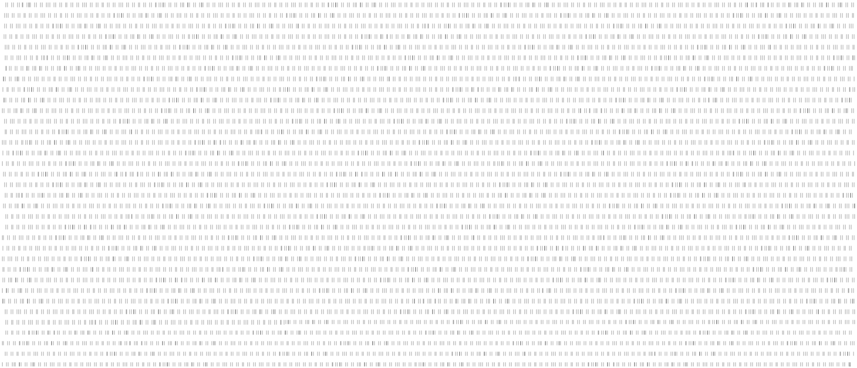
$\mathbf{s}_{7/4}$ wrapped into 100 columns

$$\mathbf{s}_{7/4} = 000000100100000010010000001001000011000000\cdots$$



$\mathbf{s}_{7/4}$ wrapped into 50847 columns

$$\mathbf{s}_{7/4} = 0000001001000000010010000000100100000110000000 \cdots$$

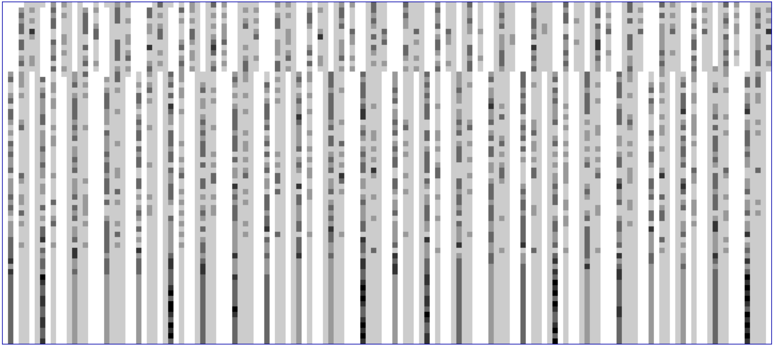


Theorem

$\mathbf{s}_{7/4} = \varphi^{\infty}(0)$ *for some* 50847*-uniform morphism* $\varphi(n) = u\,(n+2)$.

$\mathbf{s}_{5/4}$ wrapped into 144 columns

$$\mathbf{s}_{5/4} = 000011110202101001011121200001311010210130 2\cdots$$



We don't know the structure of $\mathbf{s}_{5/4}$.

Establishing the structure of $\mathbf{s}_{a/b}$

To show that $\mathbf{s}_{a/b} = \varphi^\infty(0)$:

1. Show that φ preserves $\frac{a}{b}$-power-freeness:

$$w \text{ is } \tfrac{a}{b}\text{-power-free} \implies \varphi(w) \text{ is } \tfrac{a}{b}\text{-power-free.}$$

   Since 0 is $\frac{a}{b}$-power-free, this implies $\varphi^\infty(0)$ is $\frac{a}{b}$-power-free.

2. Show that decrementing any term in $\varphi^\infty(0)$ introduces an $\frac{a}{b}$-power.

We reduce both steps to finite computations.

Proving $\frac{a}{b}$-power-freeness

We want to show that $\frac{a}{b}$-powers in $\varphi(w)$ come from $\frac{a}{b}$-powers in w.
Where can an $\frac{a}{b}$-power occur? $(xy)^{a/b} = xyx$ $\quad 1 < \frac{a}{b} < 2$

Example

Let $\varphi(n) = 000010(n+1)$ of length $k = |\varphi(n)| = 7$.
The word 000 occurs in $\varphi(w)$ at positions $\equiv 1, 2 \mod 7$.

But each word of length 4 occurs at a unique position modulo 7.
0000 0001 0010 0101 1010 0100 1000 0102 $\cdots$

We say φ locates words of length 4.

Suppose φ locates words of length k.
If $\varphi(w)$ contains an $\frac{a}{b}$-power $(xy)^{a/b} = xyx$ with $|x| \geq k$,
then k divides $|xy| = mb$ (for some m).
Assuming $\gcd(b, k) = 1$, then $k \mid m$.
Then k divides $|xyx| = ma$. By shifting, we find an $\frac{a}{b}$-power in w.

So if w is $\frac{a}{b}$-power-free, then $\varphi(w)$ does not contain long $\frac{a}{b}$-powers.

Proving lex-leastness

Show that decrementing any term in $\varphi^\infty(0)$ introduces an $\frac{a}{b}$-power.

We exploit the self-similarity of $\varphi^\infty(0)$.

Example

Let $\varphi(n) = 000010(n+1)$.
Decrementing 1 to 0 introduces the $\frac{5}{3}$-power $00000 = (000)^{5/3}$.
Decrementing $n+1$ to $c = 0$ introduces the $\frac{5}{3}$-power $00100 = (001)^{5/3}$.
Induction on c: Assume that decrementing any letter in $\varphi^\infty(0)$ to $c-1$ introduces an $\frac{a}{b}$-power ending at this $c-1$.
Let $\varphi(w)$ be a prefix of $\varphi^\infty(0)$ with last letter $n+1$. "De-substitute"; then w is a prefix of $\varphi^\infty(0)$ with last letter n.
Decrementing $n+1$ to c produces the image, under φ, of the word obtained by decrementing n to $c-1$.

So, computationally, we just need to check the base cases.

Catalogue

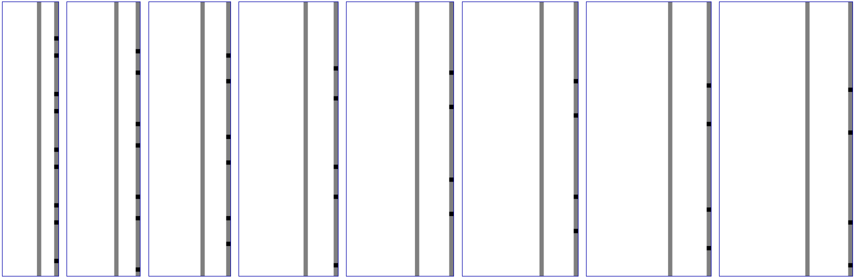
For many sequences $\mathbf{s}_{a/b}$, there is a related k-uniform morphism.

$\frac{a}{b}$	k	running time
$\frac{3}{2}$	6	
$\frac{5}{3}$	7	
$\frac{8}{5}$	733	3 seconds
$\frac{7}{4}$	50847	6 hours
$\frac{5}{4}$	?	

Question

Is this true for every $\frac{a}{b} > 1$? How is k related to $\frac{a}{b}$?

A family related to $\mathbf{s}_{5/3}$



$\mathbf{s}_{9/5}$ $\mathbf{s}_{12/7}$ $\mathbf{s}_{13/7}$ $\mathbf{s}_{16/9}$ $\mathbf{s}_{17/9}$ $\mathbf{s}_{19/11}$ $\mathbf{s}_{20/11}$ $\mathbf{s}_{21/11}$

Theorem

Let $\frac{5}{3} \leq \frac{a}{b} < 2$ *and* $\gcd(b, 2) = 1$*. Let*

$$\varphi(n) = 0^{a-1}\,1\,0^{a-b-1}\,(n+1).$$

Then $\mathbf{s}_{a/b} = \varphi^\infty(0)$*.*

We must prove $\frac{a}{b}$-power-freeness (and lex-leastness) symbolically.

Proving $\frac{a}{b}$-power-freeness symbolically

Slide length-a window through the circular word $0^{a-1}\,1\,0^{a-b-1}\,(n+1)$:

length-a factor	interval for i
$0^{a-1-i}\,1\,0^{i}$	$0 \leq i \leq a-b-1$
$0^{b-1-i}\,1\,0^{a-b-1}\,(n+1)\,0^{i}$	$0 \leq i \leq 2b-a-1$
$0^{a-b-1-i}\,1\,0^{a-b-1}\,(n+1)\,0^{2b-a+i}$	$0 \leq i \leq 2a-3b-1$
$0^{2b-a-1-i}\,1\,0^{a-b-1}\,(n+1)\,0^{a-b+i}$	$0 \leq i \leq 2b-a-1$
$0^{a-b-1-i}\,(n+1)\,0^{b+i}$	$0 \leq i \leq a-b-1$

Partition each length-a factor into xyz:

x (length $a-b$)	y (length $2b-a$)	z (length $a-b$)	interval for i
0^{a-b}	0^{2b-a}	$0^{a-b-1-i}\,1\,0^{i}$	$0 \leq i \leq a-b-1$
0^{a-b}	$0^{2b-a-1-i}\,1\,0^{i}$	$0^{a-b-1-i}\,(n+1)\,0^{i}$	$0 \leq i \leq 2b-a-1$
$0^{a-b-1-i}\,1\,0^{i}$	0^{2b-a}	$0^{2a-3b-1-i}\,(n+1)\,0^{2b-a+i}$	$0 \leq i \leq 2a-3b-1$
$0^{2b-a-1-i}\,1\,0^{2a-3b+i}$	$0^{2b-a-1-i}\,(n+1)\,0^{i}$	0^{a-b}	$0 \leq i \leq 2b-a-1$
$0^{a-b-1-i}\,(n+1)\,0^{i}$	0^{2b-a}	0^{a-b}	$0 \leq i \leq a-b-1$

Also compute factors of length $2a, 3a, \ldots, m_{\max}a$.

Check that $x \neq z$ for each factor.
We don't need a decision procedure for solvability of symbolic word equations…

Testing inequality of symbolic words

We just need to verify inequality of pairs of words we encounter.

Example

$x = 0^{a-b-1-i}\,1\,0^{i}$, $z = 0^{2a-3b-1-i}\,(n+1)\,0^{2b-a+i}$.
Since $n \geq 0$ and $\frac{5}{3} \leq \frac{a}{b} < 2$, we get $x \neq z$ by comparing prefixes.

Another heuristic: Delete the common prefix/suffix, or delete 0s, and recursively test inequality.

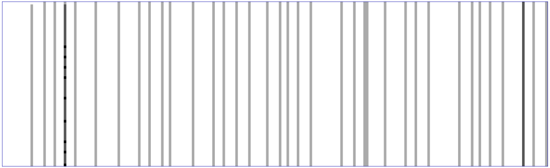
Example

$$0^{352a-621b-i-1}\,1\,0^{-51a+91b-1}\,(n+1)\,0^{i}$$
$$0^{-51a+91b-j-1}\,(n+1)\,0^{352a-621b-1}\,1\,0^{j}$$

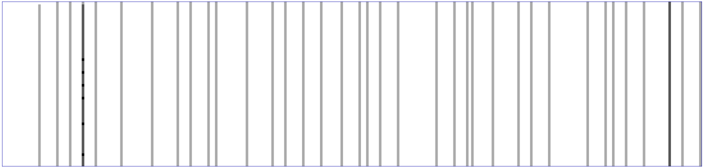
Deleting all explicit 0 letters in both words gives $1\,(n+1)$ and $(n+1)\,1$.
But these aren't unequal if $n = 0$!
Instead, look at the system of equalities of the deleted block lengths.
In this case, $-51a + 91b - 1 \neq 352a - 621b - 1$ on $\frac{30}{17} < \frac{a}{b} < \frac{53}{30}$.

A family with a transient

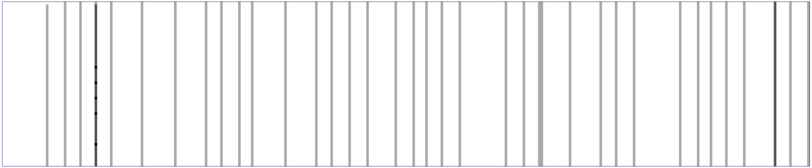
$\mathbf{s}_{17/13}$



$\mathbf{s}_{22/17}$



$\mathbf{s}_{25/19}$



The interval $\frac{9}{7} < \frac{a}{b} < \frac{4}{3}$

Theorem

Let $\frac{9}{7} < \frac{a}{b} < \frac{4}{3}$ *and* $\gcd(b, 6) = 1$. *Let*

$$\varphi(0') = 0' 0^{a-2}\, 1\, 0^{a-b-1}\, 1\, 0^{a-b-1}\, 1\, \varphi(0)$$

and

$$\begin{aligned}\varphi(n) = {} & 0^{a-b-1}\,1\,0^{2a-2b-1}\,1\,0^{-a+2b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{4a-5b-1}\,1 \\ & 0^{-a+2b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{-2a+3b-1}\,1\,0^{5a-6b-1}\,1 \\ & 0^{-2a+3b-1}\,1\,0^{4a-5b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{3a-3b-1}\,1\,0^{-2a+3b-1}\,1 \\ & 0^{a-b-1}\,1\,0^{-3a+4b-1}\,1\,0^{5a-6b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1 \\ & 0^{3a-3b-1}\,1\,0^{-2a+3b-1}\,1\,0^{4a-5b-1}\,1\,0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{2a-2b-1}\,2 \\ & 0^{a-b-1}\,1\,0^{-2a+3b-1}\,1\,0^{3a-3b-1}\,1\,0^{-2a+3b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,(n+2)\end{aligned}$$

for $n \in \mathbb{Z}_{\geq 0}$. *Then* $\mathbf{s}_{a/b} = \tau(\varphi^{\infty}(0'))$.

Other intervals

We have 30 symbolic $\frac{a}{b}$-power-free morphisms, found experimentally.

Theorem

Let $\frac{3}{2} < \frac{a}{b} < \frac{5}{3}$ *and* $\gcd(b, 5) = 1$. *The* $(5a - 4b)$*-uniform morphism*

$$\varphi(n) = 0^{a-1}\, 1\, 0^{a-b-1}\, 1\, 0^{2a-2b-1}\, 1\, 0^{a-b-1}\, (n+1)$$

is $\frac{a}{b}$*-power-free.*

Theorem

Let $\frac{6}{5} < \frac{a}{b} < \frac{5}{4}$ *and* $\frac{a}{b} \notin \{\frac{11}{9}, \frac{17}{14}\}$. *The a-uniform morphism*

$$\varphi(n) = 0^{6a-7b-1}\, 1\, 0^{-3a+4b-1}\, 1\, 0^{-8a+10b-1}\, 1\, 0^{6a-7b-1}\, (n+1)$$

is $\frac{a}{b}$*-power-free.*

Other intervals

Theorem 50. *Let a, b be relatively prime positive integers such that $\frac{10}{9} < \frac{a}{b} < \frac{29}{26}$ and $\frac{a}{b} \neq \frac{39}{35}$ and $\gcd(b, 67) = 1$. Then the $(67a - 30b)$-uniform morphism*

$$
\begin{aligned}
\varphi(n) = {} & 0^{-7a+8b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{-26a+29b-1}\,1\,0^{28a-31b-1}\,1\,0^{2a-2b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{2a-2b-1}\,1\,0^{11a-12b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{2a-2b-1}\,1\,0^{-24a+27b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{-24a+27b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1 \\
& 0^{11a-12b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{28a-31b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-7a+8b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{9a-10b-1}\,1\,0^{-7a+8b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{a-b-1}\,1\,0^{9a-10b-1}\,1 \\
& 0^{-7a+8b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{2a-2b-1}\,2\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{2a-2b-1}\,1 \\
& 0^{11a-12b-1}\,1\,0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,1\,0^{11a-12b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{11a-12b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{-24a+27b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-7a+8b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{-25a+28b-1}\,1\,0^{28a-31b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{-26a+29b-1}\,1\,0^{28a-31b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{9a-10b-1}\,1\,0^{-7a+8b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1 \\
& 0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,2 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2 \\
& 0^{a-b-1}\,1\,0^{2a-2b-1}\,1\,0^{-24a+27b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{11a-12b-1}\,1\,0^{2a-2b-1}\,1 \\
& 0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1 \\
& 0^{11a-12b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{11a-12b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-7a+8b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-7a+8b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1 \\
& 0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1\,0^{a-b-1}\,1\,0^{9a-10b-1}\,(n+1),
\end{aligned}
$$

with 279 nonzero letters, locates words of length $5a - 4b$ and is $\frac{a}{b}$-power-free.

Coverage of $\frac{a}{b}$-power-free morphisms