Algebraic power series and their automatic complexity

Eric Rowland
Hofstra University

Joint work with Manon Stipulanti and Reem Yassawi

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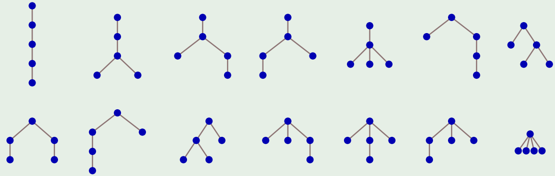
What do combinatorial sequences look like modulo p^α?

Example

Catalan numbers count plane trees with n edges:



$C(n)_{n\geq 0} = 1, 1, 2, 5, 14, 42, 132, 429, \ldots$

Modulo 2: $1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, \ldots$

$C(n)$ is odd if and only if $n + 1$ is a power of 2.
(follows from Kummer 1852)

Modulo 4: $1, 1, 2, 1, 2, 2, 0, 1, 2, 2, 0, 2, 0, 0, 0, 1, \dots$

Theorem (Eu–Liu–Yeh 2008)

For all $n \geq 0$,

$$C(n) \bmod 4 = \begin{cases} 1 & \text{if } n+1 = 2^a \text{ for some } a \geq 0 \\ 2 & \text{if } n+1 = 2^b + 2^a \text{ for some } b > a \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $C(n) \not\equiv 3 \mod 4$.

Modulo 8: $1, 1, 2, 5, 6, 2, 4, 5, 6, 6, 4, 2, 4, 4, 0, 5, \dots$

Theorem 4.2. *Let C_n be the nth Catalan number. First of all, $C_n \not\equiv_8 3$ and $C_n \not\equiv_8 7$ for any n. As for other congruences, we have*

$$C_n \equiv_8 \begin{cases} 1 & \text{if } n = 0 \text{ or } 1; \\ 2 & \text{if } n = 2^a + 2^{a+1} - 1 \text{ for some } a \geq 0; \\ 4 & \text{if } n = 2^a + 2^b + 2^c - 1 \text{ for some } c > b > a \geq 0; \\ 5 & \text{if } n = 2^a - 1 \text{ for some } a \geq 2; \\ 6 & \text{if } n = 2^a + 2^b - 1 \text{ for some } b - 2 \geq a \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Liu and Yeh (2010) determined $C(n)$ mod 16:

Theorem 5.5. *Let c_n be the n-th Catalan number. First of all, $c_n \not\equiv_{16} 3, 7, 9, 11, 15$ for any n. As for the other congruences, we have*

$$c_n \equiv_{16} \begin{cases} \left.\begin{matrix} 1 \\ 5 \\ 13 \end{matrix}\right\} & \text{if} \quad d(\alpha) = 0 \ \text{and} \quad \begin{cases} \beta \le 1, \\ \beta = 2, \\ \beta \ge 3, \end{cases} \\ \left.\begin{matrix} 2 \\ 10 \end{matrix}\right\} & \text{if} \quad d(\alpha) = 1,\ \alpha = 1 \ \text{and} \quad \begin{cases} \beta = 0 \ \text{or} \ \beta \ge 2, \\ \beta = 1, \end{cases} \\ \left.\begin{matrix} 6 \\ 14 \end{matrix}\right\} & \text{if} \quad d(\alpha) = 1,\ \alpha \ge 2 \ \text{and} \quad \begin{cases} (\alpha = 2, \beta \ge 2) \ \text{or} \ (\alpha \ge 3, \beta \le 1), \\ (\alpha = 2, \beta \le 1) \ \text{or} \ (\alpha \ge 3, \beta \ge 2), \end{cases} \\ \left.\begin{matrix} 4 \\ 12 \end{matrix}\right\} & \text{if} \quad d(\alpha) = 2 \ \text{and} \quad \begin{cases} zr(\alpha) \equiv_2 0, \\ zr(\alpha) = 1, \end{cases} \\ \quad 8 & \text{if} \quad d(\alpha) = 3, \\ \quad 0 & \text{if} \quad d(\alpha) \ge 4. \end{cases}$$

where $\alpha = (CF_2(n+1) - 1)/2$ and $\beta = \omega_2(n+1)$ (or $\beta = \min\{i \mid n_i = 0\}$).

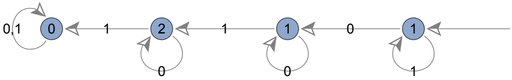
They also determined $C(n)$ mod 64.

Better framework: automatic sequences.

Automatic sequences

$s(n)_{n \geq 0}$ is *p-automatic* if there is an automaton that outputs $s(n)$ when fed the base-p digits of n (least significant digit first).

$C(n) \bmod 4$:

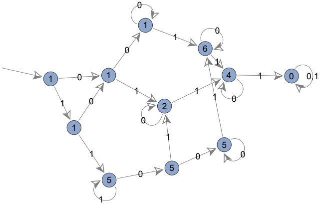


$C(9) \equiv ? \mod 4$.

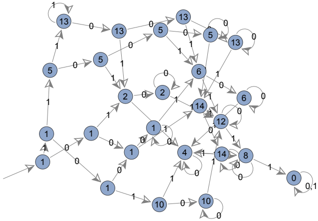
Since $9 = 1001_2$, $C(9) \equiv \boxed{2} \mod 4$.

$(C(n) \bmod 4)_{n \geq 0} = 1, 1, 2, 1, 2, 2, 0, 1, 2, 2, \ldots$ is 2-automatic.

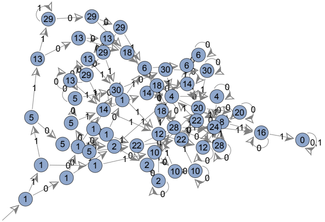
mod 8:



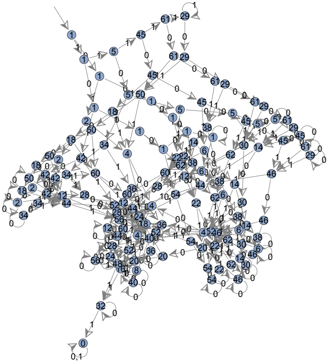
mod 16:



mod 32:



mod 64:



$(C(n) \bmod p^{\alpha})_{n\geq 0}$ is p-automatic for each $\alpha \geq 1$.

The sequence of Catalan numbers is algebraic:

$$F = \sum_{n \geq 1} C(n) x^n \quad \text{satisfies} \quad x\,(F+1)^2 - F = 0.$$

Omit $C(0) = 1 \neq 0$.

Convert to the diagonal of a rational series (Furstenberg 1967):
$P = x(y+1)^2 - y$, so

$$F = \operatorname{diag}\left(\frac{y \frac{\partial P}{\partial y}(xy, y)}{P(xy, y)/y}\right) = \operatorname{diag}\left(\frac{y - 2xy^2 - 2xy^3}{1 - x - 2xy - xy^2}\right).$$

Theorem (Denef–Lipshitz 1987)

Let $\alpha \geq 1$. Let $S(\mathbf{x}), Q(\mathbf{x}) \in \mathbb{Z}_p[\mathbf{x}]$ such that $Q(0, \dots, 0) \not\equiv 0 \mod p$. Then the coefficient sequence of $\left(\operatorname{diag} \frac{S(\mathbf{x})}{Q(\mathbf{x})}\right) \mod p^\alpha$ is p-automatic.

$\mathbb{Z}_p$ is the set of p-adic integers.

Automaton size

How big is the (unminimized) automaton for $(C(n) \bmod 2^\alpha)_{n \geq 1}$?

α	1	2	3	4	5	6	7	8	9
size	4	6	15	37	83	194	445	1034	2403
$1.4 \times 2.3^\alpha$	3.2	7.4	17.0	39.2	90.1	207.3	476.7	1096.4	2521.6

height $h = \deg_x P$
degree $d = \deg_y P$
Upper bound from the construction: $p^{p^{2(\alpha-1)}\alpha h d}$

Example

$C(n) \bmod 2^9$: $P = x(y+1)^2 - y$ $h = 1$ $d = 2$
size $\leq 2^{18 \cdot 2^{16}} = 2^{1179648}$

Why is the bound so large?

Simpler setting: finite fields.

Finite fields

Theorem (Christol 1979/1980)

A sequence $s(n)_{n\geq 0}$ of elements in $\mathbb{F}_q$ is algebraic if and only if it is q-automatic.

Two representations: polynomials and automata.

Theorem (Bridy 2017)

If the minimal polynomial P has height h and degree d, then the minimal automaton has size at most

$$(1 + o(1))q^{hd}$$

where $o(1)$ tends to 0 as any of q, h, d gets large.

Is the bound sharp? We suspect yes.

Polynomials in $\mathbb{F}_q[x, y]$ with maximum unminimized automaton size:

$q = 2$:

h	d	P	aut. size	q^{hd}	bound
1	2	$xy^2 + (x+1)y + x$	7	4	9
2	2	$x^2y^2 + (x^2+x+1)y + x^2$	14	16	25
3	2	$(x^3+x^2+1)y^2 + (x^3+1)y + x$	68	64	94
4	2	$(x^4+x+1)y^2 + (x^4+x^2+x+1)y + x$	252	256	311
5	2	$(x^5+x^3+1)y^2 + (x^5+x+1)y + x$	1052	1024	1192
6	2	$(x^6+x^5+1)y^2 + (x^6+x^2+x+1)y + x$	4062	4096	4424
7	2	$(x^7+x+1)y^2 + (x^7+x^4+x^3+x+1)y + x$	16424	16384	17288
1	3	$xy^3 + y^2 + (x+1)y + x$	11	8	18
2	3	$(x^2+x+1)y^3 + y^2 + (x^2+1)y + x^2 + x$	61	64	93
3	3	$(x^3+x+1)y^3 + y^2 + (x^3+x^2+x+1)y + x^3 + x^2$	533	512	614
4	3	$(x^4+x+1)y^3 + y^2 + (x^4+1)y + x^4 + x^3 + x$	4213	4096	4871
1	4	$(x+1)y^4 + y^2 + (x+1)y + x$	20	16	33
2	4	$(x^2+x+1)y^4 + y^3 + (x^2+x+1)y + x^2 + x$	216	256	358
3	4	$(x^3+x+1)y^4 + y^3 + (x^3+1)y + x^2 + x$	3956	4096	4870
1	5	$(x+1)y^5 + (x+1)y^2 + y + x$	37	32	67
2	5	$(x^2+x+1)y^5 + y^4 + y^3 + x^2y^2 + y + x^2 + x$	889	1024	1510
3	5	$(x^3+x^2+1)y^5 + y^4 + x^3y^2 + (x+1)y + x^3 + x^2 + x$	43913	32768	48134

$q = 3$:

h	d	P	aut. size	q^{hd}	bound
1	2	$(x+1)y^2 + y + x$	9	9	14
2	2	$(x^2+x+2)y^2 + y + x^2$	79	81	91
3	2	$(x^3+x^2+2x+1)y^2 + y + x^3 + x$	727	729	788
4	2	$(x^4+x^3+2)y^2 + y + x^4 + x$	6533	6561	6729

Can we get Bridy's bound without algebraic geometry? Yes.

Theorem (Rowland–Stipulanti–Yassawi 2023)

The minimal automaton has size at most

$$q^{hd} + q^{(h-1)(d-1)}\mathcal{L}(h,d,d) + \lfloor \log_q h \rfloor + \lceil \log_q \max(h, d-1) \rceil + 3.$$

$$P \in \mathbb{F}_q[x,y], \quad h = \deg_x P, \quad d = \deg_y P$$

Corollary (Bridy)

The minimal automaton has size at most $(1+o(1))q^{hd}$.

Step 1

size $\leq q^{(h+1)d} + 1$.

$$F = \operatorname{diag}\left(\frac{y\frac{\partial P}{\partial y}(xy,y)}{P(xy,y)/y}\right) = [y^0]\left(\frac{y\frac{\partial P}{\partial y}}{P/y}\right) \text{ sheared} \qquad \text{Let } S_0 = y\frac{\partial P}{\partial y},\ Q = P/y.$$

One Cartier operator for each digit $0, 1, \ldots, q-1$. Ex. If $q = 3$, then

$$\Lambda_1\left(a_0 + a_1 x + a_2 x^2 + \cdots\right) = a_1 + a_4 x + a_7 x^2 + \cdots.$$

$$\Lambda_r[y^0]\left(\frac{S}{Q}\right) = [y^0]\Lambda_{r,0}\left(\frac{S}{Q}\right) = [y^0]\Lambda_{r,0}\left(\frac{SQ^{q-1}}{Q^q}\right) = [y^0]\left(\frac{\Lambda_{r,0}\left(SQ^{q-1}\right)}{Q}\right)$$

Represent states by polynomials: $\lambda_{r,0}(S) := \Lambda_{r,0}\left(SQ^{q-1}\right)$.

Proposition

If $S \in \mathbb{F}_q[x,y]$ with $\deg_x S \leq h$ and $\deg_y S \leq d$, then

- $\deg_x \lambda_{0,0}(S) \leq h$ *and* $\deg_x \lambda_{r,0}(S) \leq h-1$ *for* $r \in \{1, \ldots, q-1\}$.
- $\deg_y \lambda_{r,0}(S) \leq d-1$ *for* $r \in \{0, 1, \ldots, q-1\}$.

Goal:

$$\boxed{q^{hd} + q^{(h-1)(d-1)}\mathcal{L}(h,d,d) + \lfloor \log_q h \rfloor + \lceil \log_q \max(h, d-1) \rceil + 3}$$

Step 2

size $\leq q^{hd} + |\mathrm{orb}_{\Lambda_0}(F)|$.

$\mathbb{F}_q$-vector space of polynomials with size q^{hd}:

$$W := \left\langle x^i y^j : 0 \leq i \leq h-1 \text{ and } 0 \leq j \leq d-1 \right\rangle$$

Proposition

$\lambda_{r,0}(W) \subseteq W$ *for each* $r \in \{0, 1, \ldots, q-1\}$.

Therefore every state outside $\mathrm{orb}_{\Lambda_0}(F)$ is in W.

Goal:

$$\boxed{q^{hd} + q^{(h-1)(d-1)}\mathcal{L}(h,d,d) + \lfloor \log_q h \rfloor + \lceil \log_q \max(h, d-1) \rceil + 3}$$

Step 3

$$|\mathrm{orb}_{\Lambda_0}(F)| \leq q^{(h-1)(d-1)}\mathcal{L}(h,d,d) + \lfloor \log_q h \rfloor + \lceil \log_q \max(h, d-1) \rceil + 3.$$

$\mathcal{L}(h,d,d)$ is related to the Landau function $g(n)$:

$$\begin{aligned} g(5) = \max(&\mathrm{lcm}(5), \mathrm{lcm}(4,1), \mathrm{lcm}(3,2), \mathrm{lcm}(3,1,1), \\ &\mathrm{lcm}(2,2,1), \mathrm{lcm}(2,1,1,1), \mathrm{lcm}(1,1,1,1,1)) = 6 \end{aligned}$$

We'll have 3 univariate polynomials R, with degrees $\leq h, d, d$.

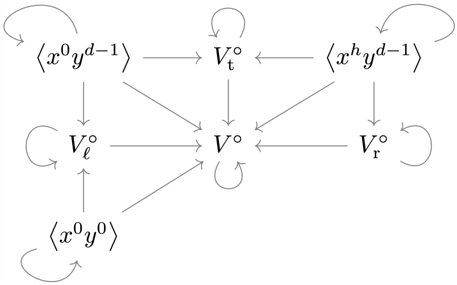
Factor each $R = R_1^{e_1} \cdots R_k^{e_k}$. $\longrightarrow$ period length $\mathrm{lcm}(\deg R_1, \dots, \deg R_k)$
and transient length $\log_q \max(e_1, \dots, e_k)$

$$\mathcal{L}(h,d,d) = \max_{\substack{1 \leq i \leq h \\ 1 \leq j \leq d \\ 1 \leq k \leq d}} \max_{\substack{\sigma_1 \in \mathrm{partitions}(i) \\ \sigma_2 \in \mathrm{partitions}(j) \\ \sigma_3 \in \mathrm{partitions}(k)}} \mathrm{lcm}(\mathrm{lcm}(\sigma_1), \mathrm{lcm}(\sigma_2), \mathrm{lcm}(\sigma_3))$$

Basis of $V \supseteq W$:

x^0y^{d-1}	x^1y^{d-1}	$\cdots$	$x^{h-1}y^{d-1}$	x^hy^{d-1}
x^0y^{d-2}	x^1y^{d-2}	$\cdots$	$x^{h-1}y^{d-2}$	x^hy^{d-2}
$\vdots$	$\vdots$	$\iddots$	$\vdots$	$\vdots$
x^0y^1	x^1y^1	$\cdots$	$x^{h-1}y^1$	x^hy^1
x^0y^0	x^1y^0	$\cdots$	$x^{h-1}y^0$	x^hy^0

Information flow under $\lambda_{0,0}$:



$\lambda_0(S) = \Lambda_0\left(SR^{q-1}\right)$ emulates $\lambda_{0,0}$ on each border.

Write $P = \sum_{i=0}^{h} x^i A_i(y) = \sum_{j=0}^{d} B_j(x) y^j$.
The 3 polynomials R are B_d, A_0, A_h, which have degrees $\leq h, d, d$.

How do we get period length $\ell = \mathrm{lcm}(\deg R_1, \dots, \deg R_k)$?

Theorem

Let $R \in \mathbb{F}_q[z]$ be a square-free polynomial with $R(0) \neq 0$ and $\deg R \geq 1$. Factor $R = cR_1 \cdots R_k$ into irreducibles. Let $\ell = \operatorname{lcm}(\deg R_1, \ldots, \deg R_k)$. Then $\lambda_0^{\ell}(S) = S$ for all $S \in \mathbb{F}_q[z]$ with $\deg S \leq \deg R$.

Proposition

The product of all monic irreducible polynomials in $\mathbb{F}_q[z]$ with degree dividing ℓ is $z^{q^\ell} - z$.

$\mathbb{F}_{q^\ell}$ is the splitting field of $z^{q^\ell} - z$ over $\mathbb{F}_q$.
Each element in $\mathbb{F}_{q^\ell}$ has a minimal polynomial over $\mathbb{F}_q$,
so multiplying all those minimal polynomials together gives $z^{q^\ell} - z$.

R divides $1 - z^{q^\ell - 1}$, say $RT = 1 - z^{q^\ell - 1}$.
Therefore the period length of $\frac{1}{R} = \frac{T}{1 - z^{q^\ell - 1}}$ divides $q^\ell - 1$.
This can be used to show $\lambda_0^{\ell}(S) = S$.

Can we use the same approach modulo p^α?

Modulo p:

Theorem (slight strengthening of Engstrom 1931)

Let $R \in \mathbb{F}_p[z]$ with $R(0) \neq 0$ and $\deg R \geq 1$.
Factor $R = cR_1^{e_1} \cdots R_k^{e_k}$ into irreducibles.
Then $\frac{1}{R}$ is periodic with period length dividing $p^{\lceil \log_p e \rceil} L$
where $e = \max_{1 \leq i \leq k} e_i$ and $L = \operatorname{lcm}_{1 \leq i \leq k} (p^{\deg R_i} - 1)$.

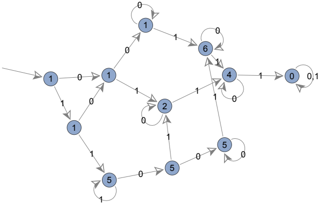
Modulo p^α:

Theorem (Engstrom 1931)

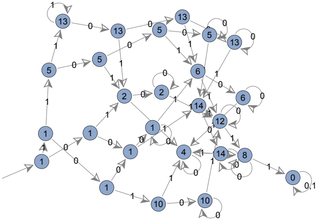
Let $R \in \mathbb{Z}/(p^\alpha \mathbb{Z})[z]$ with $r := \deg R \geq 1$ such that
the coefficients of z^0 and z^r in R are nonzero modulo p.
Then $\frac{1}{R}$ is periodic with period length dividing $p^{\alpha-1} m$
where m is the period length of $\frac{1}{R} \bmod p$.

Improved bound: $(1 + o(1)) p^{\alpha N}$ where $N = p^{2(\alpha-1)} \left(hd - \frac{1}{2}\right) + \frac{1}{2} p^{\alpha-1}$.
Singly exponential bound?

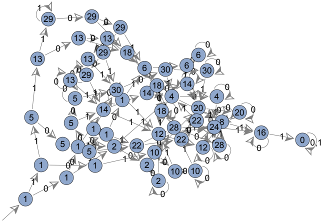
mod 8:



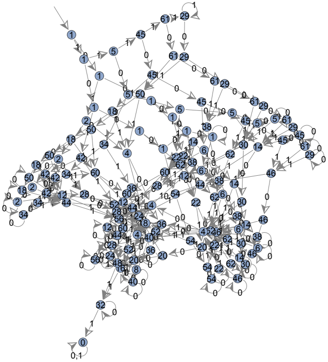
mod 16:



mod 32:



mod 64:



These automata project to each other.

So there is an inverse limit profinite automaton. Can we describe it?

$(C(n) \bmod 2)_{n \geq 0}$: $Q = (P/y \bmod 2) = xy + 1 + \frac{x}{y}$

$$\begin{aligned} S_0 &= y \\ \lambda_{0,0}(S_0) &= 0 \\ \lambda_{1,0}(S_0) &= y + 1 \end{aligned}$$

$(C(n) \bmod 4)_{n \geq 0}$: $Q = (P/y \bmod 4) = xy + 2x + 3 + \frac{x}{y}$

$$\begin{aligned} S_0 &= 2x^2y^3 + \left(2x^2 + x\right)y^2 + \left(2x^2 + 1\right)y + 2x^2 + 3x \\ \lambda_{0,0}(S_0) &= 2x^2y^2 + \left(2x^2 + 2x\right)y + 2x^2 + 2x + \frac{2x^2}{y} \\ \lambda_{1,0}(S_0) &= xy^2 + (x + 3)y + 3x + 1 + \frac{3x}{y} \end{aligned}$$

Modulo 2, these are divisible by Q.

$(C(n) \bmod 2)_{n\geq 0}$: $\qquad Q = (P/y \bmod 2) = xy + 1 + \frac{x}{y}$

$$\begin{aligned} S_0 &= y \\ \lambda_{0,0}(S_0) &= 0 \\ \lambda_{1,0}(S_0) &= y + 1 \end{aligned}$$

$(C(n) \bmod 4)_{n\geq 0}$: $\qquad Q = (P/y \bmod 4) = xy + 2x + 3 + \frac{x}{y}$

$$\begin{aligned} S_0 &= yQ + 2\left(x^2y^3 + x^2y^2 + \left(x^2 + x + 1\right)y + x^2 + x\right) \\ \lambda_{0,0}(S_0) &= 0Q + 2\left(x^2y^2 + \left(x^2 + x\right)y + x^2 + x + \frac{x^2}{y}\right) \\ \lambda_{1,0}(S_0) &= (y + 1)Q + 2\left(xy + 1 + \frac{x}{y}\right) \end{aligned}$$

Modulo 2, these are divisible by Q.

Let $D = \{0, 1, \dots, p-1\}$.

Theorem

Every state in the automaton is of the form

$$\left(T_0 + T_1 \tfrac{p}{Q} + T_2 \left(\tfrac{p}{Q}\right)^2 + \cdots + T_{\alpha-1} \left(\tfrac{p}{Q}\right)^{\alpha-1}\right) Q^{p^{\alpha-1}-1}$$

where $T_i \in D[x, y, y^{-1}]$ for each $i \in \{0, 1, \dots, \alpha-1\}$.

We can bound $\deg_x T_i$, $\deg_y T_i$, and $\operatorname{mindeg}_y T_i$.

Singly exponential upper bound:

$$p^N + |\operatorname{orb}_{\Lambda_0}(F)| = (1 + o(1))p^N$$

where $N = \frac{1}{6}\alpha(\alpha+1)((2hd-1)\alpha + hd + 1)$.

When $\alpha = 1$, we recover Bridy's $(1 + o(1))p^{hd}$ for $\mathbb{F}_p$.

References

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