A simple prime-generating recurrence

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Classification and examples

Ribenboim's conditions:

(a) $f(n)$ is the nth prime. e.g., Gandhi's formula:
$p_n = \left\lfloor 1 - \log_2 \left(-\frac{1}{2} + \sum_{d \mid \prod_{k=1}^{n-1} p_k} \frac{\mu(d)}{2^d - 1} \right) \right\rfloor$

(b) $f(n)$ is always prime, and $f(n) \neq f(m)$ for $n \neq m$.
e.g., Mills' formula: $\lfloor \theta^{3^n} \rfloor$, where $\theta = 1.3064\ldots$

(c) The set of positive values of f is equal to the set of prime numbers. e.g., multivariate prime-generating polynomials of Matijasevič and Jones et al.

But all known examples are *engineered*.

Naturally occurring functions

Are there “naturally occurring” functions that generate primes?

- Euler’s polynomial $n^2 + n + 41$ is prime for $0 \leq n \leq 39$.

The recurrence

Are there naturally occurring functions that always generate primes?

In 2003 Matthew Frank discovered the recurrence

$$f(n) = f(n-1) + \gcd(n, f(n-1)).$$

Consider the initial condition $f(1) = 7$.

First few terms

n	$\gcd(n, f(n-1))$	$f(n)$
1		7
2	1	8
3	1	9
4	1	10
5	5	15
6	3	18
7	1	19
8	1	20
9	1	21
10	1	22
11	11	33
12	3	36
13	1	37
14	1	38
15	1	39
16	1	40
17	1	41
18	1	42
19	1	43
20	1	44

n	$\gcd(n, f(n-1))$	$f(n)$
21	1	45
22	1	46
23	23	69
24	3	72
25	1	73
26	1	74
27	1	75
28	1	76
29	1	77
30	1	78
31	1	79
32	1	80
33	1	81
34	1	82
35	1	83
36	1	84
37	1	85
38	1	86
39	1	87
40	1	88

n	$\gcd(n, f(n-1))$	$f(n)$
41	1	89
42	1	90
43	1	91
44	1	92
45	1	93
46	1	94
47	47	141
48	3	144
49	1	145
50	5	150
51	3	153
52	1	154
53	1	155
54	1	156
55	1	157
56	1	158
57	1	159
58	1	160
59	1	161
60	1	162

$\gcd(n, f(n-1))$ appears to always be 1 or prime.

The sequence $\gcd(n, f(n-1))$

1, 1, 1, 5, 3, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 23, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,

47, 3, 1, 5, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,

1, 1, 1, 1, 1, 1, 1, 1, 101, 3, 1, 1, 7, 1, 1, 1, 1, 11, 3, 1, 1, 1, 1, 1, 13, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,

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1, 1, 1, 233, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,

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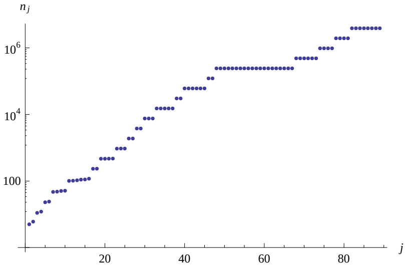
1, 1, 1, 1, 1, 1, 1, 1, 467, 3, 1, 5, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, . . .

Prime values of $\gcd(n, f(n-1))$

5, 3, 11, 3, 23, 3, 47, 3, 5, 3, 101, 3, 7, 11, 3, 13, 233, 3, 467, 3, 5, 3, 941, 3, 7, 1889, 3, 3779, 3, 7559, 3, 13, 15131, 3, 53, 3, 7, 30323, 3, 60647, 3, 5, 3, 101, 3, 121403, 3, 242807, 3, 5, 3, 19, 7, 5, 3, 47, 3, 37, 5, 3, 17, 3, 199, 53, 3, 29, 3, 486041, 3, 7, 421, 23, 3, 972533, 3, 577, 7, 1945649, 3, 163, 7, 3891467, 3, 5, 3, 127, 443, 3, 31, 7783541, 3, 7, 15567089, 3, 19, 29, 3, 5323, 7, 5, 3, 31139561, 3, 41, 3, 5, 3, 62279171, 3, 7, 83, 3, 19, 29, 3, 1103, 3, 5, 3, 13, 7, 124559609, 3, 107, 3, 911, 3, 249120239, 3, 11, 3, 7, 61, 37, 179, 3, 31, 19051, 7, 3793, 23, 3, 5, 3, 6257, 3, 43, 11, 3, 13, 5, 3, 739, 37, 5, 3, 498270791, 3, 19, 11, 3, 41, 3, 5, 3, 996541661, 3, 7, 37, 5, 3, 67, 1993083437, 3, 5, 3, 83, 3, 5, 3, 73, 157, 7, 5, 3, 13, 3986167223, 3, 7, 73, 5, 3, 7, 37, 7, 11, 3, 13, 17, 3, 19, 29, 3, 13, 23, 3, 5, 3, 11, 3, 7972334723, 3, 7, 463, 5, 3, 31, 7, 3797, 3, 5, 3, 15944673761, 3, 11, 3, 5, 3, 17, 3, 53, 3, 139, 607, 17, 3, 5, 3, 11, 3, 7, 113, 3, 11, 3, 5, 3, 293, 3, 5, 3, 53, 3, 5, 3, 151, 11, 3, 31889349053, 3, 63778698107, 3, 5, 3, 491, 3, 1063, 5, 3, 11, 3, 7, 13, 29, 3, 6899, 3, 13, 127557404753, 3, 41, 3, 373, 19, 11, 3, 43, 17, 3, 320839, 255115130849, 3, 510230261699, 3, 72047, 3, 53, 3, 17, 3, 67, 5, 3, 79, 157, 5, 3, 110069, 3, 7, 1020460705907, 3, 5, 3, 43, 179, . . .

First key observations



logarithmic plot of n_j,
the jth value of n for which
$\gcd(n, f(n-1)) \neq 1$

Ratio between clusters is very nearly 2.

Each cluster is initiated by a large prime p.

Another key observation

n	$\gcd(n, f(n-1))$	$f(n)$
1		7
2	1	8
3	1	9
4	1	10
5	5	15
6	3	18
7	1	19
8	1	20
9	1	21
10	1	22
11	11	33
12	3	36
13	1	37
14	1	38
15	1	39
16	1	40
17	1	41
18	1	42
19	1	43
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28	1	76
29	1	77
30	1	78
31	1	79
32	1	80
33	1	81
34	1	82
35	1	83
36	1	84
37	1	85
38	1	86
39	1	87
40	1	88

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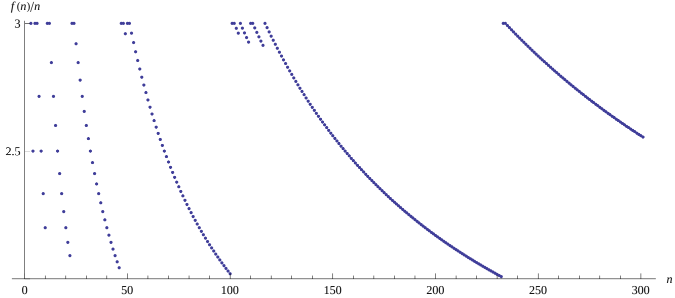
51	3	153
52	1	154
53	1	155
54	1	156
55	1	157
56	1	158
57	1	159
58	1	160
59	1	161
60	1	162

$f(n) = 3n$ whenever $\gcd(n, f(n-1)) \neq 1$.

$f(n)/n$

This suggests looking at $f(n)/n$ in general.



For $n \geq 3$, we have $2 < f(n)/n \leq 3$.

Local structure

Lemma

Let $n_1 \geq 2$. *Let* $f(n_1) = 3n_1$, *and for* $n > n_1$ *let*

$$f(n) = f(n-1) + \gcd(n, f(n-1)).$$

Let n_2 *be the smallest integer greater than* n_1 *such that* $\gcd(n_2, f(n_2 - 1)) \neq 1$. *Then*

- $\gcd(n_2, f(n_2 - 1)) = p$ *is prime,*
- p *is the smallest prime divisor of* $2n_1 - 1$,
- $n_2 = n_1 + \frac{p-1}{2}$, *and*
- $f(n_2) = 3n_2$.

This lemma provides the inductive step.

Main result

Theorem

Let $f(1) = 7$*, and for* $n > 1$ *let*

$$f(n) = f(n-1) + \gcd(n, f(n-1)).$$

For each $n \geq 2$*,* $\gcd(n, f(n-1))$ *is either* 1 *or prime.*

Shortcut

Since all 1s may be bypassed, the recurrence (with shortcut) is a naturally occurring generator of primes.

So perhaps it earns an honorable mention under Ribenboim's criterion

(b) $f(n)$ is always prime, and $f(n) \neq f(m)$ for $n \neq m$.

Is the recurrence a "magical" producer of primes?

No.
Each step requires finding the smallest prime divisor of $2n - 1$.

General case

Of course $f(1) = 7$ seems arbitrary.
Do all initial conditions produce only 1s and primes?

No.
$f(1) = 532$ produces $\gcd(18, f(17)) = 9$.
$f(1) = 801$ produces $\gcd(21, f(20)) = 21$.

Open problem

Conjecture

Let $n_1 \geq 1$ *and* $f(n_1) \geq 1$. *For* $n > n_1$ *let*

$$f(n) = f(n-1) + \gcd(n, f(n-1)).$$

Then there exists an N *such that for each* $n > N$ $\gcd(n, f(n-1))$ *is either* 1 *or prime.*

It would suffice to show that $f(n)/n$ always reaches 1, 2, or 3.