NON-REGULARITY OF $\lfloor \alpha + \log_k n \rfloor$

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Abstract. This paper presents a new proof that if k^α is irrational then the sequence $\{\lfloor \alpha + \log_k n \rfloor\}_{n \geq 1}$ is not k-regular. Unlike previous proofs, the methods used do not rely on automata or language theoretic concepts. The paper also proves the stronger statement that if k^α is irrational then the generating function in k non-commuting variables associated with $\{\lfloor \alpha + \log_k n \rfloor\}_{n \geq 1}$ is not algebraic.

Fix an integer $k \geq 2$. A sequence $\{a(n)\}_{n \geq 0}$ is *k-regular* if the $\mathbb{Z}$-module generated by the subsequences $\{a(k^e n + i)\}_{n \geq 0}$ for $e \geq 0$ and $0 \leq i < k^e$ is finitely generated. Regular sequences were introduced by Allouche and Shallit [1] and have several nice characterizations, including the following characterization as rational power series in non-commuting variables $x_0, x_1, \dots, x_{k-1}$. If $n = n_l \cdots n_1 n_0$ is the standard base-k representation of n, then let $\tau(n) = x_{n_0} x_{n_1} \cdots x_{n_l}$. The sequence $\{a(n)\}_{n \geq 0}$ is k-regular if and only if the power series $\sum_{n \geq 0} a(n)\tau(n)$ is rational. In this sense, regular sequences are analogous to constant-recursive sequences (sequences that satisfy linear recurrence equations with constant coefficients), the set of which coincides with the set of sequences whose generating functions in a single variable are rational.

The sequence $\{\lfloor \log_2(n+1) \rfloor\}_{n \geq 0}$ is an example of a 2-regular sequence, and the associated power series in non-commuting variables x_0 and x_1 is

$$\begin{aligned} f(x_0, x_1) &= \sum_{n \geq 0} \lfloor \log_2(n+1) \rfloor \tau(n) \\ &= x_1 + x_0 x_1 + 2x_1 x_1 + 2x_0 x_0 x_1 + 2x_1 x_0 x_1 + 2x_0 x_1 x_1 + 3x_1 x_1 x_1 + \cdots . \end{aligned}$$

The rational expression for this series is somewhat large; however its commutative projection is quite manageable:

$$\frac{x_1 \left(1 - x_0 - x_1 + x_0^2 + x_0 x_1\right)}{(1 - x_1)\left(1 - x_0 - x_1\right)^2}.$$

Allouche and Shallit [2, open problem 16.10] asked whether the sequence $\{\lfloor \frac{1}{2} + \log_2(n+1) \rfloor\}_{n \geq 0}$ is 2-regular. Bell [3] and later Moshe [5, Theorem 4] gave proofs that this sequence is not 2-regular. Moreover, they proved the following.

Theorem. *Let $k \geq 2$ be an integer and α be a real number. The sequence $\{\lfloor \alpha + \log_k(n+1) \rfloor\}_{n \geq 0}$ is k-regular if and only if k^α is rational.*

In this paper we prove the following theorem, which is a slightly weaker statement than the previous theorem but still establishes that if k^α is irrational then $\{\lfloor \alpha + \log_k(n+1) \rfloor\}_{n \geq 0}$ is not k-regular. Let $|\tau(n)|$ be the length of the word $\tau(n)$, i.e., $|\tau(0)| = 0$ and $|\tau(n)| = \lfloor \log_k n \rfloor + 1$ for $n \geq 1$.

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Theorem. *Let $k \geq 2$ be an integer and α be a real number. The series $f(x) = \sum_{n\geq 0} \lfloor \alpha + \log_k(n+1) \rfloor x^{|\tau(n)|}$ is rational if and only if k^α is rational.*

The proof given here is similar to Moshe's but does not require the notion of a regular language. Note that, given the associated power series

$$f(x_0, x_1, \ldots, x_{k-1}) = \sum_{n\geq 0} \lfloor \alpha + \log_k(n+1) \rfloor \, \tau(n),$$

the series in the theorem is the power series $f(x) = f(x, x, \ldots, x)$ in one variable obtained by setting $x_0 = x_1 = \cdots = x_{k-1} = x$. Therefore non-rationality of $f(x)$ implies non-regularity of $\{\lfloor \alpha + \log_k(n+1) \rfloor\}_{n\geq 0}$.

To get a sense of computing $f(x)$ in the proof of the theorem, first we examine the case where $k = 2$ and $\alpha = \frac{1}{2}$. The power series in this case is

$$\begin{aligned} f(x_0, x_1) &= \sum_{n\geq 0} \left\lfloor \frac{1}{2} + \log_2(n+1) \right\rfloor \tau(n) \\ &= x_1 + 2x_0x_1 + 2x_1x_1 + 2x_0x_0x_1 + 3x_1x_0x_1 + 3x_0x_1x_1 + 3x_1x_1x_1 + \cdots , \end{aligned}$$

and

$$\begin{aligned} f(x) &= \sum_{n\geq 0} \left\lfloor \frac{1}{2} + \log_2(n+1) \right\rfloor x^{|\tau(n)|} \\ &= x + 2x^2 + 2x^2 + 2x^3 + 3x^3 + 3x^3 + 3x^3 + 3x^4 + 3x^4 + 3x^4 + 4x^4 + \cdots \\ &= x + 4x^2 + 11x^3 + 29x^4 + 74x^5 + 179x^6 + 422x^7 + 971x^8 + 2198x^9 + \cdots \\ &= \sum_{m\geq 0} b(m) x^m. \end{aligned}$$

To write $b(m)$ in closed form, we observe how the first few terms of $\{\lfloor \frac{1}{2} + \log_2(n+1) \rfloor\}_{n\geq 0}$ gather by exponent:

$$0122\underbrace{2333}_{x^3}\underbrace{33344444}_{x^4}\underbrace{4444445555555555}_{x^5}\underbrace{55555555555556666666666666666666}_{x^6}\cdots$$

Since the length of n in binary is $|\tau(n)| = 1 + \lfloor \log_2 n \rfloor$ for $n \geq 1$, the difference $|\tau(n)| - \lfloor \frac{1}{2} + \log_2(n+1) \rfloor$ between exponent and coefficient in each term of the first sum above is either 1 or 0. In other words, the only terms that contribute to $b(m)x^m$ are of the form $(m-1)x^m$ and mx^m, so for some sequence $\{c(m)\}_{m\geq 1}$ we have

$$b(m) = (m-1)\left(c(m) - 2^{m-1}\right) + m\left(2^m - c(m)\right)$$

for $m \geq 1$. In fact $c(m)$ is the smallest value of n for which $\frac{1}{2} + \log_2(n+1) \geq m$, so $c(m) = \lfloor 2^{m-\frac{1}{2}} \rfloor$ and $b(m) = (m+1)2^{m-1} - \lfloor 2^{m-\frac{1}{2}} \rfloor$ for $m \geq 1$. Therefore

$$f(x) = \frac{1}{2(1-2x)^2} - \frac{1}{2} - \sum_{m\geq 0} \left\lfloor 2^{m-\frac{1}{2}} \right\rfloor x^m,$$

where the term $-1/2$ is needed because $b(0) = 0$.

We carry out the preceding computation more generally to prove the theorem.

Proof. Let $\operatorname{frac}(\alpha) = \alpha - \lfloor \alpha \rfloor$ denote the fractional part of α. Then

$$\begin{aligned}
f(x) &= \sum_{n \geq 0} \lfloor \alpha + \log_k (n+1) \rfloor \, x^{|\tau(n)|} \\
&= \lfloor \alpha + \log_k 1 \rfloor + \sum_{m \geq 1} \sum_{i=k^{m-1}}^{k^m - 1} \lfloor \alpha + \log_k (i+1) \rfloor \, x^m \\
&= \lfloor \alpha \rfloor + \sum_{m \geq 1} \left(\sum_{i=k^{m-1}}^{\lceil k^{m-\operatorname{frac}(\alpha)} \rceil - 2} \lfloor \alpha + \log_k (i+1) \rfloor + \sum_{i=\lceil k^{m-\operatorname{frac}(\alpha)} \rceil - 1}^{k^m - 1} \lfloor \alpha + \log_k (i+1) \rfloor \right) x^m.
\end{aligned}$$

Since

$$\lfloor \alpha + \log_k (i+1) \rfloor = \begin{cases} \lfloor \alpha \rfloor + m - 1 & \text{if } k^{m-1} + 1 \leq i + 1 \leq \lceil k^{m-\operatorname{frac}(\alpha)} \rceil - 1 \\ \lfloor \alpha \rfloor + m & \text{if } \lceil k^{m-\operatorname{frac}(\alpha)} \rceil \leq i + 1 \leq k^m, \end{cases}$$

we have

$$\begin{aligned}
f(x) &= \lfloor \alpha \rfloor + \sum_{m \geq 1} \left(k^{m-1} \left((k-1)(m + \lfloor \alpha \rfloor) + 1 \right) + 1 - \left\lceil k^{m-\operatorname{frac}(\alpha)} \right\rceil \right) x^m \\
&= \frac{(1-x)(kx + \lfloor \alpha \rfloor (1-kx))}{(1-kx)^2} + \frac{x}{1-x} + \sum_{m \geq 1} \left\lfloor -k^{m-\operatorname{frac}(\alpha)} \right\rfloor x^m.
\end{aligned}$$

The series $f(x)$ is therefore rational if and only if

$$\begin{aligned}
g(x) &= - \left\lfloor -k^{1-\operatorname{frac}(\alpha)} \right\rfloor + \left(\frac{1}{x} - k \right) \sum_{m \geq 1} \left\lfloor -k^{m-\operatorname{frac}(\alpha)} \right\rfloor x^m \\
&= \sum_{m \geq 1} \left(\left\lfloor -k^{m+1-\operatorname{frac}(\alpha)} \right\rfloor - k \left\lfloor -k^{m-\operatorname{frac}(\alpha)} \right\rfloor \right) x^m
\end{aligned}$$

is rational. The expression $\lfloor k^m y \rfloor - k \lfloor k^{m-1} y \rfloor$ is the $(-m)$th base-k digit of y, so the coefficients of $g(x)$ are the base-k digits of $\operatorname{frac}(-k^{1-\operatorname{frac}(\alpha)})$, which is rational precisely when k^α is rational.

If k^α is rational, then the coefficients of $g(x)$ are eventually periodic, so $g(x)$ and hence $f(x)$ is rational. If k^α is irrational, then $g(x)$ is not rational, since in particular $g(\frac{1}{k}) = \operatorname{frac}(-k^{1-\operatorname{frac}(\alpha)})$ is irrational; therefore $f(x)$ is not rational. □

In fact we may show something stronger: Not only does $f(x_0, x_1, \ldots, x_{k-1})$ fail to be rational when k^α is irrational, but it fails to be algebraic. Bell, Gerhold, Klazar, and Luca [4, Proposition 13] prove that if a polynomial-recursive sequence (a sequence satisfying a linear recurrence equation with polynomial coefficients) has only finitely many distinct values, then it is eventually periodic. It follows that the coefficient sequence of $g(x)$ is not polynomial-recursive, hence $g(x)$ is not algebraic, and $f(x, x, \ldots, x)$ is not algebraic.

References

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