LUCAS' THEOREM MODULO p^2

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Abstract. Lucas' theorem describes how to reduce a binomial coefficient $\binom{a}{b}$ modulo p by breaking off the least significant digits of a and b in base p. We characterize the pairs of these digits for which Lucas' theorem holds modulo p^2. This characterization is naturally expressed using symmetries of Pascal's triangle.

1. Introduction

In 1878, Lucas [11] discovered a formula for computing the residue of a binomial coefficient modulo p, where p is a prime. Namely, if $r, s \in \{0, 1, \ldots, p-1\}$ and a and b are nonnegative integers, then

$$\binom{pa+r}{pb+s} \equiv \binom{a}{b}\binom{r}{s} \mod p. \tag{1}$$

This congruence can also be written using base-p representations. Let the base-p representations of a and b be $a_\ell \cdots a_1 a_0$ and $b_\ell \cdots b_1 b_0$, where we have made them the same length by padding the shorter representation with 0s if necessary. Iterating Congruence (1) gives

$$\binom{a}{b} \equiv \binom{a_\ell}{b_\ell} \cdots \binom{a_1}{b_1}\binom{a_0}{b_0} \mod p.$$

Several variants and generalizations of Lucas' theorem are known. Meštrović [13] gives an excellent survey. In particular, it is natural to ask for Lucas-type congruences modulo higher powers of p. We refer to a congruence of the form

$$\binom{pa+r}{pb+s} \equiv \binom{a}{b}\binom{r}{s} \mod p^\alpha \tag{2}$$

where $r, s \in \{0, 1, \ldots, p-1\}$ as a *Lucas congruence*. This congruence does not hold in general, but it does hold for certain values of α, p, r, s, a, b. Even prior to Lucas' work, Babbage [1] in 1819 showed that

$$\binom{2p-1}{p-1} \equiv 1 \mod p^2 \tag{3}$$

for all $p \geq 3$; this is a Lucas congruence where $r = s = p-1$, $a = 1$, and $b = 0$. In 1862, Wolstenholme [18] showed that Babbage's congruence holds modulo p^3 if $p \geq 5$. This was generalized by Glaisher [8, page 21] in 1900 to the Lucas congruence

$$\binom{pa-1}{p-1} \equiv 1 \mod p^3 \tag{4}$$

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for all $a \geq 1$, again for $p \geq 5$. Since $a\binom{pa-1}{p-1} = \binom{pa}{p}$, this implies $\binom{pa}{p} \equiv a \mod p^3$, which itself can be generalized to the Lucas congruence

$$\binom{pa}{pb} \equiv \binom{a}{b} \mod p^3 \tag{5}$$

for $a \geq 0$, $b \geq 0$, and $p \geq 5$. Congruence (5) is often attributed to Ljunggren [3]. However, Ljunggren only considered the special case $a = pb$ and was primarily interested in the case $a = p^n, b = p^{n-1}$. The general form seems to have been first obtained by Jacobsthal in the same paper [3]. It was independently rediscovered several times, including by Kazandzidis [10] and Bailey [2]. Siong [17] also gave a proof that Congruence (5) follows from Glaisher's congruence. For $p = 2$ and $p = 3$, Congruence (5) does not hold modulo p^3 in general but does hold modulo p^2.

While each of the congruences (3)–(5) uses a single pair (r, s) of digits, some results of Bailey [2] allow these digits to be general. For every prime p, Bailey proved that

$$\binom{p^2a + r}{p^2b + s} \equiv \binom{a}{b}\binom{r}{s} \mod p^2$$

for all $r, s \in \{0, 1, \ldots, p-1\}$, $a \geq 0$, and $b \geq 0$. The equivalent form $\binom{p(pa)+r}{p(pb)+s} \equiv \binom{pa}{pb}\binom{r}{s} \mod p^2$ is a Lucas congruence. For $p \geq 5$, Bailey also proved

$$\binom{p^3a + r}{p^3b + s} \equiv \binom{a}{b}\binom{r}{s} \mod p^3.$$

These exponents 3 were subsequently increased by Davis and Webb [5]. A further extension was found by Zhao [20], and generalizations of Lucas' theorem modulo p^α for general $\alpha \geq 1$ were given by Davis and Webb [4], Granville [9], and Yassawi and the author [15, Theorem 5.3], although these results depart from the form of Congruence (2).

In this article we consider the following question. For which pairs (r, s) of base-p digits does the Lucas congruence

$$\binom{pa + r}{pb + s} \equiv \binom{a}{b}\binom{r}{s} \mod p^2$$

hold for all $a \geq 0$ and $b \geq 0$? The set of such pairs is our primary object of interest.

Notation. For each prime p, let

$$D(p) = \left\{(r, s) \in \{0, 1, \ldots, p-1\}^2 : \binom{pa + r}{pb + s} \equiv \binom{a}{b}\binom{r}{s} \mod p^2 \text{ for all } a \geq 0, b \geq 0\right\}.$$

2. Description of the set $D(p)$

Congruence (5) implies that $D(p)$ is nonempty for each prime $p \geq 5$, since $(0, 0) \in D(p)$. Computer experiments suggest that $D(p)$ contains additional pairs as well. For example, we will show that $D(3) = \{(0, 0), (2, 0), (2, 2)\}$ and $D(7) = \{(0, 0), (4, 2), (6, 0), (6, 6)\}$. The following table highlights the binomial coefficients $\binom{r}{s}$ corresponding to points $(r, s) \in D(7)$.

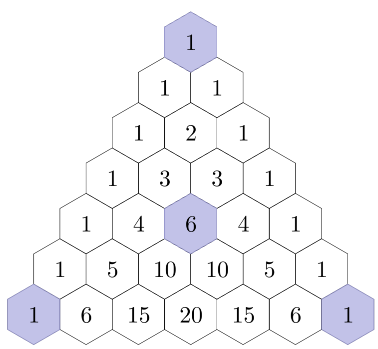
1	0	0	0	0	0	0
1	1	0	0	0	0	0
1	2	1	0	0	0	0
1	3	3	1	0	0	0
1	4	6	4	1	0	0
1	5	10	10	5	1	0
1	6	15	20	15	6	1

Our first result is that the zeros in this table do not correspond to points in $D(p)$.

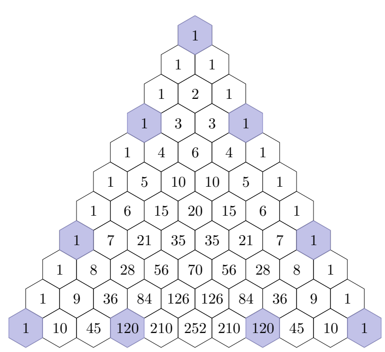
Proposition 1. *Let p be a prime. If $s > r$, then $(r, s) \notin D(p)$.*

Proof. Let $a = 1$ and $b = 0$. The binomial coefficient $\binom{pa+r}{pb+s} = \binom{p+r}{s} = \frac{(p+r)!}{s!(p+r-s)!}$ is divisible by p but not p^2. On the other hand, $\binom{a}{b}\binom{r}{s} = \binom{r}{s} = 0$ is divisible by p^2. Therefore $\binom{pa+r}{pb+s} \not\equiv \binom{a}{b}\binom{r}{s} \mod p^2$. $\square$

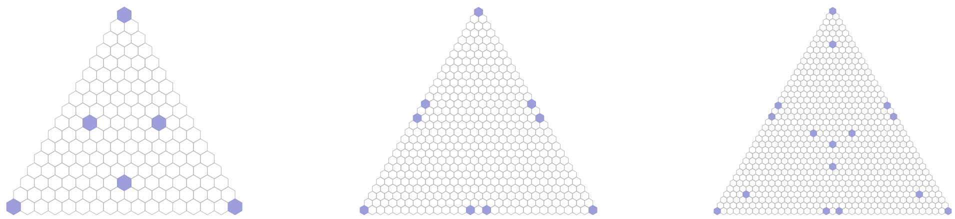
In light of Proposition 1, we omit points (r, s) where $s > r$ from the previous table. Then we shear the remaining triangle:



For $p = 11$, the set $D(11)$ contains 9 pairs of digits, arranged as follows.



For $p = 17$, $p = 29$, and $p = 37$ the pairs in $D(p)$ appear in the following locations.

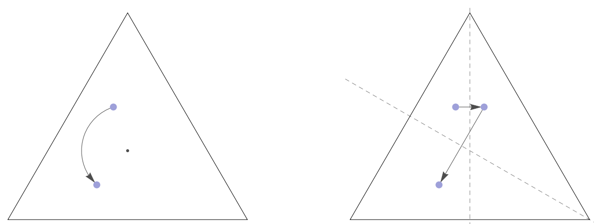


$p = 17$ $\quad$ $p = 29$ $\quad$ $p = 37$

These pictures suggest that $D(p)$ is invariant under the symmetries of the equilateral triangle!

Reflection symmetry about the vertical axis is not altogether surprising, since Pascal's triangle also exhibits this symmetry. We establish this in Proposition 5. However, the rotational symmetry of $D(p)$ is unexpected.

To identify the image of (r, s) under rotation, we use the fact that counterclockwise rotation by $120°$ is equivalent to the composition of two reflections:



The first reflection is through the vertical altitude of the triangle. This reflection maps the point (r, s) to $(r, r - s)$. The second reflection is through the altitude passing through the lower right vertex. This reflection maps (r, s) to $(p-1-r+s, s)$, as can be seen by shearing so that this altitude is horizontal. Composing these reflections shows that the rotation maps (r, s) to $(p - 1 - s, r - s)$. Therefore the three binomial coefficients visited by the orbit of (r, s) under rotation by $120°$ are

$$\binom{r}{s}, \binom{p-1-s}{r-s}, \binom{p-1-r+s}{p-1-r},$$

the third of which is equal to $\binom{p-1-r+s}{s}$.

In general, these three binomial coefficients are not equal, nor are they congruent modulo p. However, we will show in Corollary 7 that they do satisfy a congruence modulo p if we multiply them by the correct signs. Furthermore, the elements of $D(p)$ can be characterized as the pairs (r, s) for which this congruence holds not just modulo p but modulo p^2.

Theorem 2. *Let p be a prime, and let $r, s \in \{0, 1, \dots, p-1\}$. The congruence*

$$\binom{pa+r}{pb+s} \equiv \binom{a}{b}\binom{r}{s} \mod p^2$$

holds for all $a \geq 0$ and $b \geq 0$ if and only if $s \leq r$ and

$$\binom{r}{s} \equiv (-1)^{r-s}\binom{p-1-s}{r-s} \equiv (-1)^{s}\binom{p-1-r+s}{s} \mod p^2. \tag{6}$$

For certain classes of primes, $D(p)$ contains digit pairs that correspond to simple geometric points in the triangle. For example, if $p \equiv 1 \mod 3$ then the center of the triangle has integer coordinates, namely $r = \frac{2}{3}(p-1)$ and $s = \frac{1}{3}(p-1)$.

Moreover, $p \equiv 1 \mod 6$ in this case, so the coordinates r and s are even, and $1 = (-1)^{r-s} = (-1)^s$. Since the center is invariant under rotation about itself, the point (r, s) satisfies Congruence (6). Consequently, $(r, s) \in D(p)$ and we obtain the following congruence.

Corollary 3. *If* $p \equiv 1 \mod 3$*, then*

$$\binom{pa + \frac{2}{3}(p-1)}{pb + \frac{1}{3}(p-1)} \equiv \binom{a}{b}\binom{\frac{2}{3}(p-1)}{\frac{1}{3}(p-1)} \mod p^2$$

for all $a \geq 0$ *and* $b \geq 0$.

We can iterate Corollary 3 for the particular numbers $a = \frac{2}{3}(p-1)\sum_{i=0}^{\ell-1} p^i = \frac{2}{3}(p^\ell - 1)$ and $b = \frac{1}{3}(p^\ell - 1)$ whose base-p representations consist of ℓ copies of the digits $\frac{2}{3}(p-1)$ and $\frac{1}{3}(p-1)$, respectively. Therefore, if $p \equiv 1 \mod 3$ and $\ell \geq 0$, then

$$\binom{\frac{2}{3}(p^\ell - 1)}{\frac{1}{3}(p^\ell - 1)} \equiv \binom{\frac{2}{3}(p-1)}{\frac{1}{3}(p-1)}^\ell \mod p^2.$$

The value of $\binom{2(p-1)/3}{(p-1)/3}$ modulo p was studied by Jacobi, and its value modulo p^2 was shown by Yeung [19, Theorem 4.13] to be $\binom{2(p-1)/3}{(p-1)/3} \equiv -A + \frac{p}{A} \mod p^2$, where $4p = A^2 + 27B^2$ and the sign of A is chosen so that $A \equiv 1 \mod 3$.

A prime p is a *Wieferich prime* if $2^{p-1} \equiv 1 \mod p^2$. Only two such primes are known: 1093 and 3511. It will follow from the proof of Theorem 2 that p is a Wieferich prime if and only if $\{(\frac{p-1}{2}, 0), (\frac{p-1}{2}, \frac{p-1}{2}), (p-1, \frac{p-1}{2})\} \subseteq D(p)$. These digits pairs correspond to the midpoints of the three edges of the triangle. Morley [14] proved that $\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \mod p^3$ for every prime $p \geq 5$. In particular, $\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} \mod p^2$ for Wieferich primes.

An interesting question, which we do not address here, is this: What else can be said about the size of $D(p)$ as a function of p? The following table lists the elements of $D(p)$ for the first ten primes.

p	$D(p)$
2	$\{(0,0)\}$
3	$\{(0,0), (2,0), (2,2)\}$
5	$\{(0,0), (4,0), (4,4)\}$
7	$\{(0,0), (4,2), (6,0), (6,6)\}$
11	$\{(0,0), (3,0), (3,3), (7,0), (7,7), (10,0), (10,3), (10,7), (10,10)\}$
13	$\{(0,0), (8,4), (12,0), (12,12)\}$
17	$\{(0,0), (9,2), (9,7), (14,7), (16,0), (16,16)\}$
19	$\{(0,0), (12,6), (18,0), (18,18)\}$
23	$\{(0,0), (22,0), (22,22)\}$
29	$\{(0,0), (13,0), (13,13), (15,0), (15,15), (28,0), (28,13), (28,15), (28,28)\}$

Theorem 2 was suggested by an analogous result for the Apéry numbers, which are defined by $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$. Gessel [7] showed that the Apéry numbers satisfy the one-dimensional Lucas congruence $A(pn + r) \equiv A(n)A(r) \mod p$ for all $r \in \{0, 1, \ldots, p-1\}$ and all $n \geq 0$. For certain values of r, this congruence also holds modulo p^2. Gessel noticed that $A(3n + r) \equiv A(n)A(r) \mod 9$ for all $r \in \{0, 1, 2\}$. By computing an automaton for the Apéry numbers modulo 25, Yassawi and the author [15, Theorem 3.31] showed that $A(5n + r) \equiv A(n)A(r)$

mod 25 if $r \in \{0,2,4\}$. This was recently generalized to all primes [16]. Namely, the digits $r \in \{0,1,\dots,p-1\}$ for which all $n \geq 0$ satisfy

$$A(pn+r) \equiv A(n)A(r) \mod p^2$$

are precisely the digits for which $A(r) \equiv A(p-1-r) \mod p^2$. The reflection symmetry $A(r) \equiv A(p-1-r) \mod p$ was established by Malik and Straub [12, Lemma 6.2] for all $r \in \{0,1,\dots,p-1\}$. Therefore, the elements of both $D(p)$ and the analogous set for the Apéry numbers can be characterized as those for which a certain symmetry modulo p in fact holds modulo p^2.

In light of Theorem 2, it is natural to ask about digit pairs (r,s) for which the Lucas congruence holds modulo p^3 for all $a \geq 0$ and $b \geq 0$. Experiments suggest that for each prime $p \geq 5$ there are exactly three: $(0,0)$, $(p-1,0)$, and $(p-1,p-1)$. However, it is conceivable that certain primes support more. We leave this as an open question.

3. A general congruence

To prove Theorem 2, we first prove a general congruence for $\binom{pa+r}{pb+s}$ modulo p^2. Let $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ be the nth harmonic number. (In particular, the 0th harmonic number is the empty sum $H_0 = 0$.) For $r \in \{0,1,\dots,p-1\}$, the denominator of H_r is not divisible by p, so we can interpret H_n modulo p and modulo p^2.

Theorem 4. *Let p be a prime. If $0 \leq s \leq r \leq p-1$, $a \geq 0$, and $b \geq 0$, then*

$$\binom{pa+r}{pb+s} \equiv \binom{a}{b}\binom{r}{s}\left(1 + pa\left(H_r - H_{r-s}\right) + pb\left(H_{r-s} - H_s\right)\right) \mod p^2.$$

Proof. If $b > a$, then $\binom{pa+r}{pb+s} = 0 = \binom{a}{b}$, so the congruence holds. Assume $b \leq a$. By breaking a factorial into two products, we obtain

$$\begin{aligned}\binom{pa+r}{pb+s} &= \frac{(pa+r)!}{(pb+s)!(pa-pb+r-s)!} \\ &= \frac{(pa)!}{(pb)!(pa-pb)!}\,\frac{\prod_{i=1}^{r}(pa+i)}{\prod_{i=1}^{s}(pb+i)\prod_{i=1}^{r-s}(pa-pb+i)}.\end{aligned}$$

The first factor is $\binom{pa}{pb} \equiv \binom{a}{b} \mod p^2$; this is a special case of Congruence (5). In the second factor, we expand each product and collect terms by like powers of p. Namely, $\prod_{i=1}^{r}(pa+i) \equiv r! + pa\sum_{i=1}^{r}\frac{r!}{i} \mod p^2$. This gives

$$\begin{aligned}\binom{pa+r}{pb+s} &\equiv \binom{a}{b}\frac{r!}{s!(r-s)!}\,\frac{1+paH_r}{(1+pbH_s)(1+p(a-b)H_{r-s})} \mod p^2 \\ &\equiv \binom{a}{b}\binom{r}{s}(1+paH_r)(1-pbH_s)(1-p(a-b)H_{r-s}) \mod p^2 \\ &\equiv \binom{a}{b}\binom{r}{s}\left(1 + pa\left(H_r - H_{r-s}\right) + pb\left(H_{r-s} - H_s\right)\right) \mod p^2\end{aligned}$$

as desired. □

4. Symmetries of $D(p)$

In this section, we establish that $D(p)$ possesses the symmetries of the equilateral triangle. In particular, we prove Theorem 2. The reflection symmetry $\binom{a}{b} = \binom{a}{a-b}$ of Pascal's triangle is familiar. Next we show that $D(p)$ also exhibits this symmetry.

Proposition 5. *Let p be a prime. If $(r, s) \in D(p)$, then $(r, r - s) \in D(p)$.*

Proof. Let $(r, s) \in D(p)$. By Proposition 1, $s \le r$. By assumption, $\binom{pa+r}{pb+s} \equiv \binom{a}{b}\binom{r}{s}$ mod p^2 for all $a \ge 0$ and $b \ge 0$. Fix a and b. We would like to show $\binom{pa+r}{pb+r-s} \equiv \binom{a}{b}\binom{r}{r-s} \mod p^2$. There are two cases. If $b > a$, then $s < p \le p\,(b - a)$. It follows that $pa + r < pb + r - s$. Therefore $\binom{pa+r}{pb+r-s} = 0 = \binom{a}{b}\binom{r}{r-s}$, so the congruence holds. On the other hand, if $b \le a$, the reflection symmetry of Pascal's triangle gives

$$\binom{pa+r}{pb+r-s} = \binom{pa+r}{(pa+r)-(pb+r-s)} = \binom{pa+r}{p\,(a-b)+s}.$$

Since $(r, s) \in D(p)$, this implies

$$\begin{aligned}\binom{pa+r}{pb+r-s} &\equiv \binom{a}{a-b}\binom{r}{s} \mod p^2\\ &= \binom{a}{b}\binom{r}{r-s}\\ &\equiv \binom{pa}{pb}\binom{r}{r-s} \mod p^2,\end{aligned}$$

as desired. In both cases, $\binom{pa+r}{pb+r-s} \equiv \binom{a}{b}\binom{r}{r-s} \mod p^2$, so $(r, r - s) \in D(p)$. □

In addition to the reflection symmetry, the first p rows of Pascal's triangle also exhibit rotational symmetry modulo p up to sign. To see this, first we prove the following congruence modulo p^2.

Proposition 6. *Let p be a prime. If $0 \le s \le r \le p - 1$, then*

$$\binom{r}{s} \equiv (-1)^{r-s}\binom{p-1-s}{r-s}(1 + pH_r - pH_s) \mod p^2. \tag{7}$$

Proof. Similar to the proof of Theorem 4, we expand the product $(p - 1 - r)!$ and collect terms by like powers of p:

$$\begin{aligned}r!(p-1-r)! &= r!\prod_{i=r+1}^{p-1}(p-i)\\ &\equiv r!\left(\prod_{i=r+1}^{p-1}(-i) + p\,(-1)^{p-1-r}\frac{(p-1)!}{r!}\sum_{i=r+1}^{p-1}\frac{1}{-i}\right) \mod p^2\\ &= (-1)^{p-1-r}(p-1)!(1 - p\,(H_{p-1} - H_r)).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{r!(p-1-r)!}{s!(p-1-s)!} &\equiv (-1)^{r-s}\frac{1-p\,(H_{p-1}-H_r)}{1-p\,(H_{p-1}-H_s)} \mod p^2\\ &\equiv (-1)^{r-s}(1-p\,(H_{p-1}-H_r))(1+p\,(H_{p-1}-H_s)) \mod p^2\\ &\equiv (-1)^{r-s}(1+pH_r-pH_s) \mod p^2.\end{aligned}$$

This is equivalent to
$$\frac{r!}{s!} \equiv (-1)^{r-s} \frac{(p-1-s)!}{(p-1-r)!} (1 + pH_r - pH_s) \mod p^2.$$
Dividing both sides by $(r-s)!$ produces Congruence (7). □

Modulo p, we obtain the following rotational symmetry.

Corollary 7. *Let p be a prime. If $0 \leq s \leq r \leq p-1$, then*
$$\binom{r}{s} \equiv (-1)^{r-s} \binom{p-1-s}{r-s} \mod p.$$

We now use Theorem 4 and Proposition 6 to prove Theorem 2, adding a third equivalent statement. We assume $s \leq r$, since otherwise $(r, s) \notin D(p)$ by Proposition 1.

Theorem 8. *Let p be a prime, and let $0 \leq s \leq r \leq p-1$. The following are equivalent.*

1. $(r, s) \in D(p)$.
2. $H_r \equiv H_{r-s} \equiv H_s \mod p$.
3. $\binom{r}{s} \equiv (-1)^{r-s} \binom{p-1-s}{r-s} \equiv (-1)^s \binom{p-1-r+s}{s} \mod p^2$.

Proof. First we show that
$$\binom{pa+r}{pb+s} \equiv \binom{a}{b} \binom{r}{s} \mod p^2 \tag{8}$$
for all $a \geq 0$ and $b \geq 0$ if and only if $H_r \equiv H_{r-s} \equiv H_s \mod p$. By Theorem 4,
$$\binom{pa+r}{pb+s} \equiv \binom{a}{b} \binom{r}{s} (1 + pa(H_r - H_{r-s}) + pb(H_{r-s} - H_s)) \mod p^2.$$
Clearly, if $H_r \equiv H_{r-s} \equiv H_s \mod p$, then Congruence (8) holds for all $a \geq 0$ and $b \geq 0$. Conversely, assume Congruence (8) holds for all $a \geq 0$ and $b \geq 0$. Since $\binom{r}{s}$ is not divisible by p, this, along with Theorem 4, implies
$$\binom{a}{b} \equiv \binom{a}{b} (1 + pa(H_r - H_{r-s}) + pb(H_{r-s} - H_s)) \mod p^2.$$
Setting $a = 1$ and $b = 0$ shows that $H_r \equiv H_{r-s} \mod p$. Now setting $a = 1$ and $b = 1$ shows that $H_{r-s} \equiv H_s \mod p$.

Next we show the equivalence of the second and third statements. We see from Proposition 6 that $H_r \equiv H_s \mod p$ if and only if
$$\binom{r}{s} \equiv (-1)^{r-s} \binom{p-1-s}{r-s} \mod p^2.$$
Similarly, $H_r \equiv H_{r-s} \mod p$ if and only if
$$\binom{r}{r-s} \equiv (-1)^s \binom{p-1-r+s}{s} \mod p^2.$$
Since $\binom{r}{r-s} = \binom{r}{s}$, this implies that $H_r \equiv H_{r-s} \equiv H_s \mod p$ if and only if $\binom{r}{s} \equiv (-1)^{r-s} \binom{p-1-s}{r-s} \equiv (-1)^s \binom{p-1-r+s}{s} \mod p^2$. □

Theorem 8 and Proposition 5 imply that $D(p)$ is invariant under the symmetries of the equilateral triangle.

We conclude by returning to the discussion of Wieferich primes. Eisenstein [6] showed that $H_{(p-1)/2} \equiv \frac{2-2^p}{p} \mod p$ for $p \geq 3$. Therefore p is a Wieferich prime if and only if $H_{(p-1)/2} \equiv 0 \mod p$, which is equivalent to $(\frac{p-1}{2}, \frac{p-1}{2}) \in D(p)$ by Theorem 8. By rotational symmetry, p is a Wieferich prime if and only if

$$\left\{(\tfrac{p-1}{2}, 0), (\tfrac{p-1}{2}, \tfrac{p-1}{2}), (p-1, \tfrac{p-1}{2})\right\} \subseteq D(p).$$

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