Infinite Products Arising in Paperfolding

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Abstract

J.-P. Allouche recently examined two infinite products where the term is a rational function of the index $n$ raised to the term of the paperfolding sequence $\epsilon_n$. A closed form is given only for one of them. We discuss an attempt to produce the missing closed form. We give a detailed analysis of convergence and a closed form for the analogous question, where the paperfolding sequence is replaced by a periodic one.
1 Introduction

The paperfolding sequence $\epsilon_n$ is defined by the rules

$$
\begin{align*}
\epsilon_{2n} &= (-1)^n \\
\epsilon_{2n+1} &= \epsilon_n.
\end{align*}
$$

(1)

The first few values are $\{1, 1, -1, 1, 1\}$. For fixed $a \in \mathbb{N}$, the rules (1) determine all subsequences of the form

$$
\{\epsilon_{2^an+b} : a \in \mathbb{N}, 0 \leq b < 2^a\}
$$

(2)
in terms of constants, $\{\epsilon_n\}$ and $\{(-1)^n\}$. For example, when $a = 2$,

$$
\begin{align*}
\epsilon_{4n} &= 1, \quad \epsilon_{4n+1} = \epsilon_{2n} = (-1)^n, \quad \epsilon_{4n+2} = (-1)^{2n+1} = -1, \quad \epsilon_{4n+3} = \epsilon_{2n+1} = \epsilon_n.
\end{align*}
$$

(3)

The work presented here is motivated by results given by Allouche [1]. In particular, the evaluation

$$
B = \prod_{n=1}^{\infty} \left(\frac{2n}{2n+1}\right)^{\epsilon_n} = \frac{1}{8\sqrt{2\pi}} \Gamma\left(\frac{1}{4}\right)^2
$$

(4)
is obtained using the auxiliary product

$$
A = \prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2}\right)^{\epsilon_n}.
$$

(5)

Indeed, the identity

$$
AB = \frac{1}{2} \prod_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{\epsilon_n}
$$

(6)
is split according to the parity of $n$ and (1) yields

$$
AB = \frac{1}{2} A \prod_{n=1}^{\infty} \left(\frac{2n}{2n+1}\right)^{-1}
$$

(7)

The non-vanishing of $A$ gives

$$
B = \frac{1}{2} \prod_{n=0}^{\infty} \frac{(4n+4)(4n+3)}{(4n+5)(4n+2)}.
$$

(8)

A classical result expressing such products in terms of the gamma function gives the value of $B$. Observe that the value of $A$ does not come from this formulation. A search for a closed form for $A$ was the motivation for the results presented here.

An early evaluation of an infinite product was produced by Wallis in his representation

$$
\prod_{n=1}^{\infty} \frac{(2n)(2n)}{(2n-1)(2n+1)} = \frac{\pi}{2}.
$$

(9)
The history of this discovery appears in Osler [7]. The literature contains a variety of infinite product evaluations. For instance,
\[
\prod_{n=1}^\infty \left(1 + \frac{1}{F_{2^n+1}}\right) = \frac{3}{\varphi} \quad \text{and} \quad \prod_{n=1}^\infty \left(1 + \frac{1}{L_{2^n+1}}\right) = 3 - \varphi,
\]
(10)
is given by Sondow [9]. Here \(F_n, L_n\) are the Fibonacci (Lucas) numbers and \(\varphi = \frac{1}{2}(\sqrt{5} + 1)\) is the golden ratio.

The value of infinite products usually involves classical constants of analysis. For instance, Borwein [3] evaluates the function
\[
D(x) = \lim_{n \to \infty} \prod_{k=1}^{2n+1} \left(1 + \frac{x}{k}\right)^{(-1)^{k+1}k}
\]
(11)
as a generalization of the values
\[
\prod_{n=1}^\infty \left(1 + \frac{2}{n}\right)^{(-1)^{n+1}n} = \frac{\pi}{2e} \quad \text{and} \quad \prod_{n=1}^\infty \left(1 + \frac{2}{n}\right)^{(-1)^{n+1}n} = \frac{6}{\pi e}
\]
(12)
established by Melzak [6]. Some exact evaluations are given in terms of the constant
\[
A_1 = \exp\left(\frac{1}{4} - \int_0^\infty \frac{e^{-s}}{s^3} \left(1 - \frac{s^2}{2} + \frac{s^2}{12} - \frac{s}{e^s - 1}\right) \, ds\right)
\]
(13)
and the Catalan constant
\[
G = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2}.
\]
(14)
Examples include
\[D(1) = \frac{A_1^6}{2^{1/6} \sqrt{\pi}} \quad \text{and} \quad D\left(\frac{1}{4}\right) = \frac{2^{1/6} \sqrt{\pi} A_1^3}{\Gamma\left(\frac{1}{4}\right)} e^{G/\pi}.
\]
(15)
Other types of products involving gamma factors have recently been analyzed by Chamberland and Straub [4].

The question considered here deals with the evaluation of products of the form
\[
\Psi(R, s) = \prod_{n=1}^\infty R(n)^{s_n}.
\]
(16)
Here \(R\) is a rational function and \(s\) is an automatic sequence (as studied by Allouche [1]). Examples include periodic sequences taking values in the alphabet \(+1, -1\) or \(k\)-automatic sequences: a sequence \(\{s_n : n \geq 0\}\) is \(k\)-automatic if the set of subsequences \(\{s_{kn+\ell} : n \geq 0\}\) with \(j \geq 0, \ell \in [0, k^j - 1]\) is finite. More information about such sequences appears in [2].
The main example discussed here is the \textit{paperfolding sequence} $\epsilon_n$ defined in (1). Splitting the evaluation of a product into even and odd indices leads, in the special case of a rational function of degree 1, to the identity
\begin{equation}
\prod_{n=0}^{\infty} \left( \frac{\alpha n + \beta}{\gamma n + \delta} \right)^{\epsilon_n} = \prod_{n=0}^{\infty} \left( \frac{2\alpha n + \beta}{2\gamma n + \delta} \right)^{(-1)^n} \times \prod_{n=0}^{\infty} \left( \frac{2\alpha n + \alpha + \beta}{2\gamma n + \gamma + \delta} \right)^{\epsilon_n}.
\end{equation}
(17)
The exponent $(-1)^n$ appearing in the first product on the right is a periodic sequence of period length 2. This motivates the evaluation of products with terms of the form $R(n)^{M_n}$, where $M_n$ is a periodic sequence. This is the topic of Sections 2–4.

Section 2 discusses the convergence of the product
\begin{equation}
\mathcal{P}(R, 1) = \prod_{n=0}^{\infty} R(n),
\end{equation}
(18)
where
\begin{equation}
R(z) = \frac{(z + a_1) \cdots (z + a_d)}{(z + b_1) \cdots (z + b_d)}.
\end{equation}
(19)
This section reviews the elementary arguments showing that convergence in (18) is equivalent to $R(n) \to 1$ as $n \to \infty$ and $\mathcal{S}(R) = 0$. Here
\begin{equation}
\mathcal{S}(R) = \sum_{b \in R^{-1}(\infty)} b - \sum_{a \in R^{-1}(0)} a.
\end{equation}
(20)
The value of $\mathcal{P}(R, 1)$ is then given by
\begin{equation}
\prod_{n=0}^{\infty} \left( \frac{n + a_1}{n + b_1} \right) \cdots \left( \frac{n + a_d}{n + b_d} \right) = \prod_{k=1}^{d} \frac{\Gamma(b_k)}{\Gamma(a_k)}.
\end{equation}
(21)
Section 3 discusses the convergence of products $\mathcal{P}(R, M)$, where $R$ is a rational function and $M$ is a periodic sequence of period length 2. Section 4 extends the results to any periodic sequence, with special emphasis on period lengths 3 and 4. Section 5 considers some infinite products related to the paperfolding sequence, and Section 6 considers a generalization to certain $k$-automatic sequences. An alternative proof of the evaluation of Allouche’s product $B$ is presented and a new form of the product $A$ is given. The question of existence of a closed form for $A$ remains open.

\section{Convergence of infinite products}

This section considers the simplest type of product (16): $R$ is a given rational function and $s_n \equiv 1$. The data for the rational function is a sequence of complex numbers $\{a_k\}$ and $\{b_k\}$ where $a_k$, $b_k$ are not 0 nor a negative integer. The convergence of the partial finite products
\begin{equation}
\mathcal{P}_r(R, 1) = \prod_{n=1}^{r} \frac{(n + a_1) \cdots (n + a_d)}{(n + b_1) \cdots (n + b_r)}
\end{equation}
(22)
is examined first.

**Theorem 1.** The infinite product

\[ \Psi(R, 1) = \prod_{n=1}^{\infty} \frac{(n + a_1) \cdots (n + a_d)}{(n + b_1) \cdots (n + b_r)} \]  

(23)

converges if and only if \( d = r \) and \( a_1 + \cdots + a_d = b_1 + \cdots + b_r \); that is, \( R(n) \to 1 \) and \( \mathcal{S}(R) = 0 \).

**Proof.** The convergence of a product \( \prod (1 + u_k) \) is equivalent to the convergence of the series \( \sum u_k \). Therefore \( u_k \to 0 \) is a necessary condition for convergence. This implies \( d = r \). On the other hand

\[ \frac{(n + a_1) \cdots (n + a_d)}{(n + b_1) \cdots (n + b_r)} = 1 + (a_1 + \cdots + a_d - b_1 - \cdots - b_r) \frac{1}{n} + O(1/n^2), \]  

(24)

and the second condition on the parameters \( a_k, b_k \) is now clear. \( \square \)

The next question is the evaluation of the limiting product. The motivation for the final result is this: consider the problem of producing a function \( h(z) \) with zeros at a prescribed sequence \( \{z_n\} \). This is elementary if the sequence is finite: the solution is simply given as

\[ P(z) = \prod_{n=1}^{N} \left(1 - \frac{z}{z_j}\right) \]  

(25)

when \( z_j \neq 0 \). On the other hand, if the sequence is infinite, convergence issues might appear. For instance, if one would like to have a function that vanishes precisely at the negative integers, then the natural first attempt

\[ P_1(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) \]  

(26)

fails to converge. To fix this, introduce an exponential correction and form the partial products

\[ P_{2,N}(z) = e^{z \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}\right)} \prod_{n=1}^{N} \left(1 + \frac{z}{n}\right) e^{-z/n} \]  

(27)

\[ = e^{z(E_1(N) + \ln N)} \prod_{n=1}^{N} \left(1 + \frac{z}{n}\right) e^{-z/n}, \]

with

\[ E_1(N) = 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \ln N. \]  

(28)
The limit
\[ \gamma = \lim_{N \to \infty} E_1(N) \]  
(29)
is the famous Euler constant. Therefore, the modified product
\[ P_{2,N}(z) := e^{z E_1(N)} \prod_{n=1}^{N} \left(1 + \frac{z}{n}\right) e^{-z/n} \]  
(30)
has a limit as \( N \to \infty \). The infinite product has zeros at the negative integers. It turns out to be convenient to write an infinite product with poles at the negative integers and also to include 0 as a pole. This yields the classical gamma function \( \Gamma(z) \). The functional equation \( \Gamma(z + 1) = z \Gamma(z) \) is used to simplify the result.

**Theorem 2.** The infinite product representation of the gamma function is given by
\[ \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} = e^{\gamma z} \Gamma(z + 1). \]  
(31)

It is now easy to write the value of the infinite product
\[ \mathcal{P}(R, 1) = \prod_{n=1}^{\infty} \frac{(n + a_1) \cdots (n + a_d)}{(n + b_1) \cdots (n + b_r)} \]  
(32)
in Theorem 1. Start with
\[ \mathcal{P}(R, 1) = \prod_{n=1}^{\infty} \frac{(1 + b_1/n)^{-1} e^{b_1/n} \cdots (1 + b_r/n)^{-1} e^{b_r/n}}{(1 + a_1/n)^{-1} e^{a_1/n} \cdots (1 + a_d/n)^{-1} e^{a_d/n}} \]  
(33)
and observe that the added exponential terms amount to 1. Passing to the limit in (33) gives
\[ \mathcal{P}(R, 1) = \prod_{k=1}^{d} \frac{\Gamma(b_k + 1)}{\Gamma(a_k + 1)}. \]  
(34)

To simplify the form of the result, shift \( n \) to \( n + 1 \) in (32) to produce the following result.

**Theorem 3.** Let \( a_k, b_k \in \mathbb{C} \) none of which are 0 or negative integers. Assume
\[ a_1 + \cdots + a_d = b_1 + \cdots + b_d. \]  
(35)
Then
\[ \prod_{n=0}^{\infty} \frac{(n + a_1) \cdots (n + a_d)}{(n + b_1) \cdots (n + b_d)} = \prod_{k=1}^{d} \frac{\Gamma(b_k)}{\Gamma(a_k)}. \]  
(36)
3 The first example: Sequences of period length 2

This section considers products of the form

$$\mathfrak{P}(R, M) := \prod_{n=0}^{\infty} R(n)^{M_n}$$

where $M_n = (-1)^n$.

Start with the representation

$$R(z) = C \frac{(z + a_1) \cdots (z + a_d)}{(z + b_1) \cdots (z + b_r)}.$$  \hfill (38)

The partial products of $\mathfrak{P}(R, s)$ are

$$\prod_{n=0}^{N} R(n)^{(-1)^n} = \prod_{n=0}^{\lfloor N/2 \rfloor} \frac{R(2n)}{R(2n+1)} \times \begin{cases} 1, & \text{if } N \text{ is odd;} \\ R(N+1), & \text{if } N \text{ is even.} \end{cases}$$ \hfill (39)

The first factor on the right in (39) is connected to the product $\mathfrak{P}(R_1, 1)$, where

$$R_1(z) = \frac{R(2z)}{R(2z+1)}.$$ \hfill (40)

Its convergence is decided by Theorem 1. It is clear that the product on the left-hand side of (39) converges if and only if both factors on the right converge separately.

In particular, if $\mathfrak{P}(R, M)$ converges, then $\lim_{n \to \infty} R(n) = 1$ and it must be that $C = 1$ in (38). To complete the discussion, it suffices to determine conditions under which $\mathfrak{P}(R_1, 1)$ is finite. The rational function (40) factors as

$$R_1(z) = \frac{(2z + a_1) \cdots (2z + a_d)}{(2z + b_1) \cdots (2z + b_r)} \times \frac{(2z + 1 + b_1) \cdots (2z + 1 + b_r)}{(2z + 1 + a_1) \cdots (2z + 1 + a_d)},$$ \hfill (41)

with $d + r$ zeros at

$$-\frac{1}{2} a_1, \ldots, -\frac{1}{2} a_d, -\frac{1}{2} (1 + b_1), \ldots, -\frac{1}{2} (1 + b_r)$$ \hfill (42)

and $d + r$ poles at

$$-\frac{1}{2} b_1, \ldots, -\frac{1}{2} b_r, -\frac{1}{2} (1 + a_1), \ldots, -\frac{1}{2} (1 + a_d).$$ \hfill (43)

Since $R_1(z) \to 1$ as $z \to \infty$, convergence in (39) requires the relation

$$\sum_{k=1}^{d} a_k + \sum_{k=1}^{r} (1 + b_k) = \sum_{k=1}^{r} b_k + \sum_{k=1}^{d} (1 + a_k).$$ \hfill (44)

This is equivalent to the condition $d = r$. 

7
The value of $\Psi(R, M)$ is obtained from Theorem 3 as

$$
\Psi(R, M) = \Psi(R_1, 1) = \prod_{k=1}^{d} \frac{\Gamma\left(\frac{b_k}{2}\right)\Gamma\left(\frac{a_k}{2}\right)}{\Gamma\left(1 + \frac{b_k}{2}\right)\Gamma\left(1 + \frac{a_k}{2}\right)}.
$$

(45)

This is simplified using the duplication formula for the gamma function to obtain

$$
\prod_{k=1}^{d} \frac{\Gamma\left(\frac{b_k}{2}\right)\Gamma\left(\frac{1+a_k}{2}\right)}{\Gamma\left(1 + \frac{b_k}{2}\right)\Gamma\left(1 + \frac{a_k}{2}\right)} = 2^{(b_1-a_1)+\cdots+(b_d-a_d)} \prod_{k=1}^{d} \frac{\Gamma\left(\frac{b_k}{2}\right)\Gamma\left(a_k\right)}{\Gamma\left(\frac{a_k}{2}\right)\Gamma\left(b_k\right)}.
$$

(46)

The discussion above is summarized in the next statement.

**Theorem 4.** Let $R(z)$ be a rational function and $M_n = (-1)^n$. Then $\Psi(R, M)$ converges if and only if $R(z) \to 1$ as $z \to \infty$. If

$$
R(z) = \prod_{k=1}^{d} \frac{(z + a_k)}{(z + b_k)} \quad \text{and} \quad \mathcal{R}(R) = \sum_{k=1}^{d} b_k - \sum_{k=1}^{d} a_k,
$$

then

$$
\Psi(R, M) = 2^{\mathcal{R}(R)} \prod_{k=1}^{d} \frac{\Gamma\left(\frac{b_k}{2}\right)\Gamma\left(a_k\right)}{\Gamma\left(\frac{a_k}{2}\right)\Gamma\left(b_k\right)}.
$$

(48)

**Example 5.** Let $R(z) = (20z + 5)/(20z + 4)$. The convergence conditions are satisfied and Theorem 4 gives

$$
\prod_{n=0}^{\infty} \left(\frac{20n + 5}{20n + 4}\right)^{(-1)^n} = \frac{\Gamma\left(\frac{1}{10}\right)\Gamma\left(\frac{5}{8}\right)}{\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{1}{10}\right)}.
$$

(49)

Mathematica 9.0 does not evaluate the original product, but it does give the right-hand side of (49) for

$$
\Psi(R_1, 1) = \prod_{n=0}^{\infty} \frac{80n^2 + 58n + 6}{80n^2 + 58n + 5}.
$$

(50)

**Example 6.** The infinite product

$$
\Psi(R, s) = \prod_{n=0}^{\infty} \left(\frac{2 \alpha n + \beta}{2 \gamma n + \delta}\right)^{(-1)^n}
$$

(51)

encountered in the paperfolding product (17) converges if and only if $\alpha = \gamma$. The product is then

$$
\Psi(R, s) = \prod_{n=0}^{\infty} \left(\frac{n + 2\gamma}{n + 2\alpha}\right)^{(-1)^n} = 2^{2(v-u)} \frac{\Gamma^2(u)\Gamma(2v)}{\Gamma^2(v)\Gamma(2u)},
$$

(52)

with $u = \delta/4\alpha$ and $v = \beta/4\alpha$. 
4 Convergence for periodic sequences

This section discusses the issue of convergence of the product

$$\Psi(R, M) = \prod_{n=0}^{\infty} R(n)^{M_n}$$

where \(\{M_n\}\) is a periodic sequence of period length \(\ell\) of elements of the alphabet \(\{+1, -1\}\).

**Notation.** The results are expressed in terms of

\[M^+ = \{i: M_i = +1 \text{ and } 0 \leq i \leq \ell - 1\} = \{i_1, i_2, \ldots, i_{|M^+|}\}\]

\[M^- = \{j: M_j = -1 \text{ and } 0 \leq j \leq \ell - 1\} = \{j_1, j_2, \ldots, j_{|M^-|}\},\]

and the period length is \(\ell = |M^+| + |M^-|\).

The rational function is written as

\[R(n) = C(n + a_1) \cdots (n + a_d) \bigg/ (n + b_1) \cdots (n + b_r)\]

with \(a_s, b_t \notin \{0, -1, 2, \ldots\}\) and

\[\mathcal{G}(R) = \sum_{t=1}^{r} b_t - \sum_{s=1}^{d} a_s.\]

The partial product associated with \(\Psi(R, M)\) is

\[
\prod_{n=0}^{N} R(n)^{M_n} = \prod_{k=0}^{[N/\ell]} \prod_{i \in M^+} R(k\ell + i)^{M_i} \prod_{j \in M^-} R(k\ell + j)^{M_j} \prod_{n=\ell\lfloor N/\ell \rfloor + 1}^{N} R(n)^{M_n} \\
= \prod_{k=0}^{[N/\ell]} \prod_{i \in M^+} R(k\ell + i) \prod_{j \in M^-} R(k\ell + j) \prod_{n=\ell\lfloor N/\ell \rfloor + 1}^{N} R(n)^{M_n},
\]

the last product being empty if \(N\) is a multiple of the period length \(\ell\). An elementary argument shows that the convergence of \(\Psi(R, M)\) requires the convergence of both products in (57). The first product, which would lead to an expression of the form \(\Psi(R_1, 1)\) for a new rational function \(R_1\) is labeled the *main term*. The second product is called the *tail product*. We analyze its convergence first.

The tail product is defined by

\[P_{N, \ell}(M) = \prod_{n=\ell\lfloor N/\ell \rfloor + 1}^{N} R(n)^{M_n}.\]
Its convergence implies \( R(n) \to 1 \) as \( n \to \infty \). Observe that \( P_{N,\ell}(M) = 1 \) if \( N \equiv 0 \pmod{\ell} \). On the other hand, in the case \( N \equiv 1 \pmod{\ell} \), one obtains

\[
P_{N,\ell}(M) = R(N)^{M_N} = R(N)^{M_1},
\]

since \( M_N = M_1 \) by periodicity. Therefore, the convergence of \( \Psi(R, M) \) requires \( R(N) \to 1 \) for \( N \equiv 1 \pmod{\ell} \). Similarly, if \( N \equiv 2 \pmod{\ell} \),

\[
P_{N,\ell}(M) = R(N-1)^{M_N-1}R(N)^{M_N} = R(N-1)^{M_1}R(N)^{M_2}.
\]

The convergence of \( \Psi(R, M) \) already implies \( R(N-1) \to 1 \) since \( N-1 \equiv 1 \pmod{\ell} \). This time it is required that \( R(N) \to 1 \). Iterating this argument it follows that \( R(N) \to 1 \) for \( N \equiv j \pmod{\ell} \) for any residue class \( j \). This gives the next result.

**Proposition 7.** Assume \( \Psi(R, M) \) converges. Then \( \lim_{n \to \infty} R(n) = 1 \).

The limiting value of the main term is \( \Psi(R_1, 1) \), where

\[
R_1(n) = \frac{R(\ell n + i_1) \cdots R(\ell n + i_{|M^+=1|})}{R(\ell n + j_1) \cdots R(\ell n + j_{|M^-|})}.
\]  \hspace{1cm} (59)

The ingredients entering into the convergence of \( \Psi(R_1, 1) \) are discussed in the next result. We assume the condition \( R(n) \to 1 \).

**Proposition 8.** Let \( M_* = |M^+| - |M^-| \) and assume \( \Psi(R_1, 1) \) converges. Then \( \lim_{n \to \infty} R_1(n) = 1 \) and

\[
\ell \mathcal{S}(R_1) = M_* \mathcal{S}(R).
\]  \hspace{1cm} (60)

**Proof.** The behavior of \( R_1(n) \) as \( n \to \infty \) comes directly from that of \( R \). The identity (60) is a direct computation.

Combining these propositions gives the following.

**Theorem 9.** Let \( R \) be a rational function satisfying \( \lim_{n \to \infty} R(n) = 1 \) with zeros and poles of \( R \) are outside \( \{0, -1, -2, \ldots \} \). There are two cases.

1. Assume \( M_* \neq 0 \). Then \( \Psi(R, M) \) converges if and only if \( \mathcal{S}(R) = 0 \).

2. Assume \( M_* = 0 \). Then \( \Psi(R, M) \) always converges.

For a general periodic sequence, the value of the product \( \Psi(R, M) \) is given by the following.
Theorem 10. Let $R(n)$ be a rational function written in the form

$$R(n) = \frac{(n + a_1) \cdots (n + a_d)}{(n + b_1) \cdots (n + b_d)}$$

(61)

with $a_i, b_j \notin \{0, -1, -2, \ldots \}$. Let $\{M_n\}$ be a periodic sequence of $\pm 1$ with period length $\ell$. Assume the product

$$\mathcal{P}(R, M) = \prod_{n=0}^{\infty} R(n)^{M_n}$$

(62)

converges. Then

$$\mathcal{P}(R, M) = \ell^{\mathcal{O}(R)} \prod_{1 \leq s \leq d} \frac{\Gamma(a_s)}{\Gamma(b_s)} \prod_{i \in M^+} \frac{\Gamma^2 \left( \frac{b_s+i}{\ell} \right)}{\Gamma^2 \left( \frac{a_s+i}{\ell} \right)}.$$

(63)

Proof. Splitting the product according to its residues modulo $\ell$ gives

$$\prod_{n=1}^{\infty} R(n)^{M_n} = \prod_{n=0}^{\infty} \prod_{i \in M^+} \frac{R(\ell n + i)}{R(\ell n + j)}$$

$$= \prod_{i \in M^+} \prod_{n=0}^{\infty} \left( n + \frac{a_1+i}{\ell} \right) \cdots \left( n + \frac{a_d+i}{\ell} \right) \left( n + \frac{b_1+j}{\ell} \right) \cdots \left( n + \frac{b_d+j}{\ell} \right).$$

The products may be expressed in terms of the gamma function to obtain

$$\prod_{n=1}^{\infty} R(n)^{M_n} = \prod_{i \in M^+} \frac{\Gamma \left( \frac{b_1+i}{\ell} \right) \cdots \Gamma \left( \frac{b_d+i}{\ell} \right) \Gamma \left( \frac{a_1+i}{\ell} \right) \cdots \Gamma \left( \frac{a_d+i}{\ell} \right)}{\Gamma \left( \frac{a_1+i}{\ell} \right) \cdots \Gamma \left( \frac{a_d+i}{\ell} \right) \Gamma \left( \frac{b_1+i}{\ell} \right) \cdots \Gamma \left( \frac{b_d+i}{\ell} \right)}$$

(64)

and the result is simplified using Gauss’ multiplication formula

$$(2\pi)^{\frac{\ell-1}{2}} \ell^{\frac{1}{2}-\ell z} \Gamma(\ell z) = \prod_{j=0}^{\ell-1} \Gamma \left( z + \frac{j}{\ell} \right).$$

(65)

Take $z = a_s/\ell$ to produce

$$\left(2\pi\right)^{\frac{\ell-1}{2}} \ell^{1/2-a_s} \Gamma(a_s) = \Gamma \left( \frac{a_s}{\ell} \right) \Gamma \left( \frac{a_s+1}{\ell} \right) \cdots \Gamma \left( \frac{a_s+\ell-1}{\ell} \right)$$

$$= \Gamma \left( \frac{a_s+i_1}{\ell} \right) \cdots \Gamma \left( \frac{a_s+i_{\lceil M^+ \rceil}}{\ell} \right) \Gamma \left( \frac{a_s+j_1}{\ell} \right) \cdots \Gamma \left( \frac{a_s+j_{\lceil M^+ \rceil}}{\ell} \right)$$

since every residue modulo $\ell$ appears exactly once in the sets $M^+$ and $M^-$. It follows that

$$\prod_{j \in M^-} \Gamma \left( \frac{a_s+j}{\ell} \right) = \frac{\left(2\pi\right)^{(\ell-1)/2} \ell^{1/2-a_s} \Gamma(a_s)}{\prod_{i \in M^+} \Gamma \left( \frac{a_s+i}{\ell} \right)},$$

(66)

for $1 \leq s \leq d$. A similar result holds for $b_s$. Replacing in (64) concludes the proof. \qed

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**Example 11.** Consider the sequence \( \{1, -1, -1\} \), where the bar indicates the fundamental period; that is,

\[
M_n = \begin{cases} 
1, & \text{if } n \equiv 0 \pmod{3}; \\
-1, & \text{if } n \equiv 1, 2 \pmod{3}.
\end{cases}
\] (67)

Therefore \( M^+ = \{0\}, M^- = \{1, 2\} \) so that \( M_* = -1 \). Theorem 9 states that the convergence of \( \Psi(R_1, 1) \) is equivalent to \( \mathcal{S}(R) = 0 \). Take \( R(z) = \frac{(z + 1)(z + 3)}{(z + 2)^2} \). The conditions for convergence of \( \Psi(R, M) \) are satisfied, and its value is

\[
\prod_{n=0}^{\infty} \left( \frac{(n+1)(n+3)}{(n+2)^2} \right)^{M_n} = \frac{\Gamma(1) \Gamma^2 \left( \frac{2}{3} \right) \Gamma(3) \Gamma^2 \left( \frac{2}{3} \right)}{\Gamma(2) \Gamma^2 \left( \frac{1}{3} \right) \Gamma(2) \Gamma^2 \left( \frac{3}{3} \right)} = 2 \cdot \frac{\Gamma^4 \left( \frac{2}{3} \right)}{\Gamma^2 \left( \frac{1}{3} \right)} = \frac{3}{2\pi^2} \Gamma^6 \left( \frac{2}{3} \right). \] (68)

by Theorem 10.

**Example 12.** Let \( R(z) = \frac{(z + 2)(z + 3)}{(z + 1)(z + 4)} \) and \( M = \{1, 1, 1, -1\} \). Then \( M^+ = \{0, 1, 2\} \) and \( M^- = \{3\} \). Thus \( M_* \neq 0 \). The product \( \Psi(R, M) \) converges by Theorem 9, and Theorem 10 gives

\[
\prod_{n=0}^{\infty} \left( \frac{(n+2)(n+3)}{(n+1)(n+4)} \right)^{M_n} = \frac{1}{24\pi} \Gamma^4 \left( \frac{1}{4} \right). \] (69)

### 5 The paperfolding sequence

The paperfolding sequence is defined by the rules

\[
\epsilon_{2n} = (-1)^n \text{ and } \epsilon_{2n+1} = \epsilon_n. \] (70)

Allouche [1] considered the products

\[
A = \prod_{n=0}^{\infty} \left( \frac{2n+1}{2n+2} \right)^{\epsilon_n} \text{ and } B = \prod_{n=1}^{\infty} \left( \frac{2n}{2n+1} \right)^{\epsilon_n}, \] (71)

and proved

\[
B = \frac{\Gamma \left( \frac{1}{4} \right)^2}{8\sqrt{2\pi}}. \] (72)

The closed-form evaluation of \( A \) remains an open problem.

The goal of this section is to present a new proof of (72) and to present an alternative product expression for \( A \). Observe that

\[
\prod_{n=0}^{\infty} \left( \frac{an+b}{cn+d} \right)^{\epsilon_n} = \prod_{n=0}^{\infty} \left( \frac{2an+b}{2cn+d} \right)^{(-1)^n} \times \prod_{n=0}^{\infty} \left( \frac{2an+(a+b)}{2cn+(c+d)} \right)^{\epsilon_n}. \] (73)
The convergence of the first product requires \( a = c \) and its value has been obtained in Theorem 4 as
\[
\prod_{n=0}^{\infty} \left( \frac{2an+b}{2cn+d} \right)^{(-1)^n} = 2^{d/2c-b/2a} \Gamma^2 \left( \frac{d}{2c} \right) \Gamma \left( \frac{b}{2a} \right) \Gamma \left( \frac{d}{2c} \right) .
\] (74)

Iterating this procedure converts the second factor in (73) into
\[
\prod_{n=0}^{\infty} \left( \frac{2an+(a+b)}{2cn+(c+d)} \right)^{\epsilon_n} = \prod_{n=0}^{\infty} \left( \frac{4an+(a+b)}{4cn+(c+d)} \right)^{(-1)^n} \times \prod_{n=0}^{\infty} \left( \frac{4an+(3a+b)}{4cn+(3c+d)} \right)^{\epsilon_n} .
\] (75)

The first product on the right-hand side of (75) converges and Theorem 4 gives
\[
\prod_{n=0}^{\infty} \left( \frac{4an+(a+b)b}{4cn+(c+d)} \right)^{(-1)^n} = 2^{d/4c-b/4a} \Gamma^2 \left( \frac{c+d}{8c} \right) \Gamma \left( \frac{a+b}{4a} \right) \Gamma \left( \frac{c+d}{4c} \right) .
\] (76)

Now observe that
\[
\frac{c+d}{8c} = \frac{1}{4} + \frac{d-c}{8c}
\] (77)
so (76) can be written as
\[
\prod_{n=0}^{\infty} \left( \frac{4an+(a+b)b}{4cn+(c+d)} \right)^{(-1)^n} = 2^{d/4c-b/4a} \Gamma^2 \left( \frac{1}{4} + \frac{d-c}{8c} \right) \Gamma \left( \frac{1}{2} + \frac{b-a}{2a} \right) .
\] (78)

Repeated application of this process gives
\[
\prod_{n=0}^{\infty} \left( \frac{an+b}{cn+d} \right)^{\epsilon_n} = 2^{(d/c-b/a) \sum_{k=1}^{N} 1/2^k} \times \prod_{k=2}^{N} \left[ \frac{\Gamma^2 \left( \frac{1}{4} + \frac{d-c}{2^k} \right) \Gamma \left( \frac{1}{2} + \frac{b-a}{2^k-1} \right)}{\Gamma^2 \left( \frac{1}{4} + \frac{b-a}{2^k} \right) \Gamma \left( \frac{1}{2} + \frac{d-c}{2^k-1} \right)} \times \prod_{n=0}^{\infty} \left( \frac{2^N an + a(2N-1) + b}{2^N cn + c(2N-1) + d} \right) \right]^{\epsilon_n} .
\] (79)

A direct argument shows that the last product converges to 1 when \( N \to \infty \). This completes the proof of the next statement.

**Theorem 13.** The infinite product associated with the paperfolding sequence is given by
\[
\prod_{n=0}^{\infty} \left( \frac{an+b}{cn+d} \right)^{\epsilon_n} = 2^{(d/c-b/a)} \prod_{k=2}^{\infty} \left[ \frac{\Gamma^2 \left( \frac{1}{4} + \frac{d-c}{2^k} \right) \Gamma \left( \frac{1}{2} + \frac{b-a}{2^k-1} \right)}{\Gamma^2 \left( \frac{1}{4} + \frac{b-a}{2^k} \right) \Gamma \left( \frac{1}{2} + \frac{d-c}{2^k-1} \right)} \right] .
\] (80)
The product appearing in Theorem 13 does not seem to admit a simple closed form for general choice of the parameters $a, b, d$ (recall that $a = c$ is required for the convergence of the product). Such a closed form is obtained in the special situation where the factors telescope. This occurs when $2d = a + b$. The next corollary (equivalent to a theorem of Allouche [1, Theorem 1]) gives such a closed form, with $\alpha = d/a$. In that situation

$$\prod_{k=2}^{N} \frac{\Gamma^2 \left( \frac{1}{4} + \frac{d-c}{2k} \right)}{\Gamma^2 \left( \frac{1}{4} + \frac{b-a}{2k} \right)} \to \frac{\Gamma^2 \left( \frac{1}{4} \right)}{\Gamma^2 \left( \frac{a}{2} - \frac{1}{4} \right)}$$ (81)

and

$$\prod_{k=2}^{N} \frac{\Gamma \left( \frac{1}{2} + \frac{b-a}{2k} \right)}{\Gamma \left( \frac{1}{2} + \frac{d-c}{2k} \right)} \to \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma \left( \frac{1}{2} \right)}.$$ (82)

**Corollary 14.** A special case of the paperfolding product is given by

$$\prod_{n=0}^{\infty} \left( \frac{n + 2\alpha - 1}{n + \alpha} \right)^{\epsilon_n} = 2^{1-\alpha} \frac{\Gamma^2 \left( \frac{1}{4} \right) \Gamma \left( \alpha - \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right) \Gamma^2 \left( \frac{\alpha}{2} - \frac{1}{4} \right)}.$$ (83)

**Example 15.** Take $\alpha = 3$ to obtain

$$\prod_{n=0}^{\infty} \left( \frac{n + 5}{n + 3} \right)^{\epsilon_n} = 3.$$ (84)

**Example 16.** The infinite product $B$ in (71) comes by taking the limit as $\alpha \to \frac{1}{2}$. Indeed, write (83) as

$$\prod_{n=1}^{\infty} \left( \frac{n + 2\alpha - 1}{n + \alpha} \right)^{\epsilon_n} = \frac{\alpha}{\alpha - \frac{1}{2}} \frac{\Gamma^2 \left( \frac{1}{4} \right) \Gamma \left( \alpha - \frac{1}{2} \right)}{2^\alpha \Gamma \left( \frac{1}{2} \right) \Gamma^2 \left( \frac{\alpha}{2} - \frac{1}{4} \right)}.$$ (85)

The limit

$$\lim_{x \to 0} \frac{\Gamma(x)}{x \Gamma^2(x/2)} = \frac{1}{4}$$ (86)

gives

$$\prod_{n=1}^{\infty} \left( \frac{2n}{2n + 1} \right)^{\epsilon_n} = \frac{\Gamma^2 \left( \frac{1}{4} \right)}{8\sqrt{2\pi}},$$ (87)

confirming (72).

**Example 17.** The method described above does not produce a closed form for the product $A$ in (71). A direct use of the expression in Theorem 13 gives

$$A = \prod_{n=0}^{\infty} \left( \frac{2n + 1}{2n + 2} \right)^{\epsilon_n} = \sqrt{2} \prod_{k=2}^{\infty} \left( \frac{\Gamma \left( \frac{1}{4} \right)}{\Gamma \left( \frac{1}{4} - \frac{1}{2k+1} \right)} \right)^2 \times \frac{\Gamma \left( \frac{1}{2} - \frac{1}{2\pi} \right)}{\Gamma \left( \frac{1}{2} \right)}.$$ (88)
Iterating the duplication formula for the gamma function yields the so-called Knar formula [5, volume 1, page 6, formula 6]

$$\Gamma(1 + z) = 2^{2z} \prod_{k=1}^{\infty} \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1}{2} + \frac{z}{2^k} \right)$$ (89)

and $z = -\frac{1}{2}$ gives

$$\prod_{k=2}^{\infty} \frac{\Gamma \left( \frac{1}{2} - \frac{1}{2^k} \right)}{\Gamma \left( \frac{1}{2} \right)} = 2\sqrt{\pi}. \quad (90)$$

Then (88) becomes

$$A = \prod_{n=0}^{\infty} \left( \frac{2n + 1}{2n + 2} \right)^{\epsilon_n} = 2\sqrt{2\pi} \prod_{k=3}^{\infty} \left( \frac{\Gamma \left( \frac{1}{4} \right)}{\Gamma \left( \frac{1}{4} - \frac{1}{2^k} \right)} \right)^2. \quad (91)$$

The authors have been unable to reduce this any further.

6 Generalization to certain $k$-automatic sequences

This section extends the results on the paperfolding sequence to certain $k$-automatic sequences. As usual, let $R(z)$ be a rational function written in the form

$$R(z) = \frac{(z + a_1) \cdots (z + a_d)}{(z + b_1) \cdots (z + b_d)} \quad (92)$$

and assume that $a_i$ and $b_j$ are not in $\{0, -1, -2, \ldots \}$. Consider the case in which $M_n$ is a 3-automatic sequence defined by the rules

$$M_{3n} = q_0(n), \quad \quad \quad M_{3n+1} = q_1(n), \quad \quad \quad M_{3n+2} = M_n, \quad (93)$$

where $q_j$ takes values in $\{+1, -1\}$ and $q_j(n)$ is periodic of period length $\ell_j$. Now split the product according to residues modulo 3 to produce

$$\prod_{n=0}^{\infty} R(n)^{M_n} = \prod_{n=0}^{\infty} R(3n)^{M_{3n}} \times \prod_{n=0}^{\infty} R(3n + 1)^{M_{3n+1}} \times \prod_{n=0}^{\infty} R(3n + 2)^{M_{3n+2}}$$

$$= \prod_{n=0}^{\infty} R(3n)^{q_0(n)} \times \prod_{n=0}^{\infty} R(3n + 1)^{q_1(n)} \times \prod_{n=0}^{\infty} R(3n + 2)^{M_n}. \quad \text{The convergence and values of the first two products are provided by Theorem 9 and Theorem 10.}$$
Assume the convergence of the product

$$
P_0 = \prod_{n=0}^{\infty} R(3n)^{q_0(n)}. \quad (94)$$

Theorem 9 shows that this happens if $|q_0| = 0$, where $|q_0|$ is the number of +1 minus the number of −1 in one period. In the remaining case, it is required that $\mathcal{S}(R(3z)) = 0$, where $\mathcal{S}(R)$ is defined in (20). The exact form of the product is obtained from Theorem 10 which yields, with $R_0(z) = R(3z)$,

$$
P_0 = \mathcal{P}(R_0, q_0) = \ell_0^{\mathcal{S}(R_0)} \prod_{1 \leq s \leq d} \Gamma(a_s/3) \prod_{n \in q_0^+} \Gamma^2 \left( \frac{b_s + 3i}{3\ell_0} \right). \quad (95)$$

A similar process gives an analytic formula for the second product. Repeating the previous process yields a decomposition of the third product as

$$
\prod_{n=0}^{\infty} R(n)^{M_n} = \prod_{n=0}^{\infty} R(9n + 2)^{q_0(n)} \times \prod_{n=0}^{\infty} R(9n + 5)^{q_1(n)} \times \prod_{n=0}^{\infty} R(9n + 8)^{M_n}.
$$

As before, the first two products have an explicit analytic expression and the last one has to be split again.

This process can be iterated to obtain a formula for the original product. For simplicity, the results are given for $R(z)$ a rational function of degree 1 and only in the case in which all the periodic pieces $q_i(n)$ have a period length that is a power of a fixed even integer. In this situation, the final formula can be simplified.

**Theorem 18.** Let $R(z) = \frac{z + b}{z + d}$, with $b, d \in \mathbb{R}^+$ and let $M_n$ be a $k$-automatic sequence satisfying the rules

$$
M_{kn} = q_0(n) \quad M_{kn+1} = q_1(n) \quad \vdots \quad M_{kn+k-2} = q_{k-2}(n) \quad M_{kn+k-1} = M_n.
$$

Assume there is an even integer $L$ such that each sequence $q_i(n)$ is a periodic sequence of period length $L_i = L^{a_i}$ some power of $L$. In addition, assume that $|q_i^+| = |q_i^-|$ for all $0 \leq i \leq k - 2$. Then

$$
\mathcal{P}(R, M) = \prod_{n=0}^{\infty} R(n)^{M_n}. \quad (96)
$$
converges. Moreover, if $d = \frac{b + k - 1}{k}$ the product in (96) can be evaluated as

$$\prod_{n=0}^{\infty} R(n)^{M_n} = \prod_{i=0}^{k-2} \left( L_i^{k-2} \frac{\Gamma\left( \frac{b+i}{k} \right)}{\Gamma\left( \frac{i}{k} \right)} \prod_{j \in q_i^+} \frac{\Gamma^2 \left( \frac{i+j}{L_i k} + \frac{j}{L_i} \right)}{\Gamma^2 \left( \frac{b+i}{L_i k} + \frac{j}{L_i} \right)} \right). \quad (97)$$

Note that the paperfolding sequence satisfies the hypothesis of the theorem. In this case $k = 2$ and $q_0(n) = (-1)^n$, and $L = 2$. The rational function is

$$R(n) = \frac{n + b}{n + \frac{b+1}{2}}$$

and (97) reduces to the result of Allouche. The idea of the proof is the argument presented in the case of the 3-automatic sequence above. Complete details may be found in [8].

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