A NATURAL BIJECTION FOR CONTIGUOUS PATTERN AVOIDANCE IN WORDS

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Abstract. Two words p and q are avoided by the same number of length-n words, for all n, precisely when p and q have the same set of border lengths. Previous proofs of this theorem use generating functions but do not provide an explicit bijection. We give a bijective proof for all pairs p, q that have the same set of proper borders, establishing a natural bijection from the set of words avoiding p to the set of words avoiding q.

1. Introduction

Combinatorialists have studied pattern avoidance in multiple contexts. In this paper, we are interested in the avoidance of contiguous patterns in words. We say that a word w *avoids* a word p if w does not contain a contiguous occurrence of p. We refer to the word p as a *pattern*. For example, the word 010 avoids the pattern 00 but does not avoid 10. Let $\mathbb{N}$ denote the set of non-negative integers.

Definition 1.1. Let p and q be two words on a finite alphabet Σ. Define

$$A_n(p) = \{w \in \Sigma^n : \text{w avoids } p\}.$$

The words p and q are *avoidant-equivalent* if $|A_n(p)| = |A_n(q)|$ for all $n \in \mathbb{N}$.

This notion of equivalence is analogous to Wilf equivalence for non-contiguous permutation patterns, which has been studied extensively. When two permutation patterns are avoided by the same number of permutations, researchers seek a bijective explanation. See for example the survey by Claesson and Kitaev [1] of bijections between permutations that avoid 321 and permutations that avoid 132.

Analogously, when two words p, q are avoidant-equivalent, we would like a natural bijection from $A_n(p)$ to $A_n(q)$ for each n, since this provides a combinatorial explanation for the equivalence and therefore a deeper understanding of the relationship between these two structures. One can obtain a trivial bijection from $A_n(p)$ to $A_n(q)$ by first sorting the two sets lexicographically and then mapping the ith word of $A_n(p)$ to the ith word of $A_n(q)$. However, this bijection is mostly arbitrary; it assumes we already know that $|A_n(p)| =$

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$|A_n(q)|$, and it is computationally intensive, since computing the image of a particular word requires first computing the complete sets $A_n(p)$ and $A_n(q)$. In this paper, we establish a more natural bijection for many pairs of patterns. In particular, our bijection provides a new proof of avoidant-equivalence for these pairs.

A sufficient condition for two patterns to be avoidant-equivalent has essentially been known since the work of Solov'ev [8], who determined the expected time required for a pattern p to appear in a word built randomly letter by letter. Solov'ev showed that the expected time depends only on the lengths of the borders of p.

Definition 1.2. Let p be a word. A non-empty word x is a *border* of p if x is both a prefix and a suffix of p. Let

$$b(p) = \{|x| : x \text{ is a border of } p\}.$$

We call $b(p)$ the *border length set* of p. A non-empty word x is a *proper border* of p if x is a border of p and $x \neq p$.

Example. Let $\Sigma = \{0, 1\}$. The borders of $p = 0110$ are 0 and 0110. These borders can be thought of as the ways p can overlap itself:

```
   0110      0110      0110     0110
0110       0110      0110       0110
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The border length set is $b(0110) = \{1, 4\}$. The only proper border of p is 0.

It follows from the paper of Solov'ev, and more explicitly from the work of Guibas and Odlyzko [3], that if $b(p) = b(q)$ then p and q are avoidant-equivalent. Moreover, Guibas and Odlyzko give a method for computing the generating function of the number of words avoiding a pattern (or set of patterns). Let $k = |\Sigma|$ be the size of the alphabet, let $l = |p|$, and define the polynomial $B(x) = \sum_{i \in b(p)} x^{l-i}$. Then

$$\sum_{n \geq 0} |A_n(p)| x^n = \frac{B(x)}{(1 - kx)B(x) + x^l}. \tag{1}$$

This generating function was obtained by Kim, Putcha, and Roush [5] and Zeilberger [9]. It can also be obtained by the Goulden–Jackson cluster method [2]; see the treatment by Noonan and Zeilberger [6] for a friendly introduction.

Example. For the word $p = 0110$, the border length set is $b(p) = \{1, 4\}$. For $q = 1011$, we have $b(q) = \{1, 4\}$ as well. Therefore $b(p) = b(q)$, and the series expansion of $\frac{x^3+1}{1-2x+x^3-x^4}$ gives the sizes of both $A_n(p)$ and $A_n(q)$ for all $n \in \mathbb{N}$. In particular, p and q are avoidant-equivalent.

The main result of this paper (Theorem 3.1) is the following. Suppose p and q are words on a finite alphabet Σ. If the set of proper borders of p is equal to the set of proper borders of q, then the map ϕ_L, which is defined in Section 2 and iteratively replaces occurrences of q with p, is a bijection from $A_n(p)$ to $A_n(q)$ for all n. Note that here the condition is that the sets of proper borders themselves are equal, as opposed to the sets of border lengths.

For words on the binary alphabet $\Sigma = \{0, 1\}$, there are 103764 pairs of length-10 avoidant-equivalent patterns, and our theorem provides a bijection for 71058 of these pairs. Additionally, there are two types of trivial bijections — left–right reversal and permutations of Σ. Compositions of all these bijections provide bijections for 103460 pairs, which is 99.7% of avoidant-equivalent pairs of length-10 patterns. See Section 3.1 and Table 1 for more data. The smallest pair of avoidant-equivalent patterns on $\{0, 1\}$ for which we do not have a natural bijection is 0010010 and 0110110, which have a border length set of $\{1, 4, 7\}$.

Example. Let $p = 1001$ and $q = 1101$. Since p and q have the same set of proper borders, namely $\{1\}$, the replacement function ϕ_L forms a bijection from $A_n(p)$ to $A_n(q)$. We would also like a bijection from $A_n(0110)$ to $A_n(q)$, since $b(0110) = \{1\} = b(q)$. The patterns 0110 and q do not have the same set of proper borders, since 0 is a border of 0110 but is not a border of q. However, if we let σ be the letter permutation function, which replaces 0's with 1's and 1's with 0's, then σ forms a bijection from $A_n(0110)$ to $A_n(p)$. Therefore the composition $\phi_L \circ \sigma$ is a bijection from $A_n(0110)$ to $A_n(q)$.

We mention that the sufficient condition $b(p) = b(q)$ for the patterns p, q to be avoidant-equivalent is also necessary. This follows from the rational generating function in Equation (1), which provides a linear recurrence satisfied by $|A_n(p)|$. Namely, let $k = |\Sigma|$ and $l = |p|$ again, and let $s(n) = |A_n(p)|$. Then

$$s(n) = k\, s(n-1) - s(n-l) + \sum_{\substack{i \in b(p) \\ i \neq l}} \Big(k\, s(n+i-l-1) - s(n+i-l) \Big).$$

Using this recurrence, one can show that if $b(p) \neq b(q)$ then the sequence $(|A_n(p)|)_{n \geq 0}$ first differs from $(|A_n(q)|)_{n \geq 0}$ at

$$n = \begin{cases} \min(|p|, |q|) & \text{if } |p| \neq |q| \\ 2|p| - \max\big(b(p) \triangle b(q)\big) & \text{if } |p| = |q| \end{cases}$$

where $\triangle$ denotes symmetric difference. Therefore, the patterns p and q are avoidant-equivalent if and only if $b(p) = b(q)$.

In Section 2, we define replacement functions ϕ_L and ϕ_R. Section 3 is dedicated to proving the main theorem, namely that ϕ_L establishes a bijection from $A_n(p)$ to $A_n(q)$ under the condition that the proper borders of p and q are identical.

2. Replacement functions

In this section, we define the function ϕ_L that, under certain conditions, gives a bijection $\phi_L\colon A_n(p) \to A_n(q)$ in Section 3. The general idea is to systematically replace each occurrence of q in a word with p. We accomplish this with an iterative replacement process. We will define ϕ_L to take a p-avoiding word and scan from left to right looking for occurrences of q. If it finds q, it replaces the first occurrence of q with p and then starts the left-to-right scan over. The replacement process ends when no more q's remain. We will prove in Lemma 2.3 below that this process terminates.

In the following definitions, we assume that we have two patterns p and q such that $b(p) = b(q)$. In particular, $|p| = |q|$. Let $f^k(w)$ be the word obtained by iteratively applying k iterations of the function f to w.

Definition 2.1. For a given p-avoiding word w, the *single scan function* L replaces the leftmost q in w with p. If no q exists, L acts as the identity function. Define $\phi_L(w) = L^i(w)$, where i is the least non-negative integer such that $L^i(w)$ contains no q's.

Even though we are scanning left to right, a replacement in one position can be followed by a replacement to its left, as the following example shows.

Example. Let $p = 011$ and $q = 001$. The iterative replacement process of ϕ_L on the word $0001001 \in A_7(p)$ is as follows:

$$\begin{aligned} 0001001 &\overset{L}{\mapsto} 0011001 \\ &= 0011001 \overset{L}{\mapsto} 0111001 \\ &\qquad = 0111001 \overset{L}{\mapsto} 0111011 \\ &\qquad\qquad = 0111011. \end{aligned}$$

Thus, $\phi_L(0001001) = 0111011$. We have $0111011 \in A_7(q)$ as desired.

To prove that ϕ_L forms a bijection from $A_n(p)$ to $A_n(q)$, we will prove that there exists a natural inverse function ϕ_R. To this end, we define the functions R and ϕ_R, which are built to undo their counterparts L and ϕ_L.

Definition 2.2. For a given q-avoiding word w, the *single scan function* R replaces the rightmost p in w with q. If no p exists, R acts as the identity function. Define $\phi_R(w) = R^j(w)$, where j is the least non-negative integer such that $R^j(w)$ contains no p's.

Example. Using $p = 011$ and $q = 001$ as in the previous example, one checks that $\phi_R(0111011) = 0001001$, so $\phi_R(\phi_L(0001001)) = 0001001$.

Lemma 2.3. *Let p and q be equal-length patterns such that $p \neq q$, and let $n \in \mathbb{N}$. For every $w \in A_n(p)$, we have $\phi_L(w) \in A_n(q)$.*

Proof. Since $p \neq q$, either $p < q$ or $p > q$ lexicographically. Assume $p < q$, since the other case is analogous. If w contains q, then $L(w) < w$. Therefore, iteratively applying L produces lexicographically smaller words until the image no longer contains q. Since there are only finitely many length-n words on Σ, this happens after finitely many steps, at which point we have a word in $A_n(q)$. □

For a word w, we define $\overline{w}$ to be the reverse of w. Let $\overline{L}$ be the function that replaces the leftmost occurrence of $\overline{p}$ in a word with $\overline{q}$. Similarly, let $\overline{R}$ be the function that replaces the rightmost $\overline{q}$ in a word with $\overline{p}$.

Definition 2.4. We now define the functions $\overline{\phi_L}\colon A_n(\bar{q}) \to A_n(\bar{p})$ and $\overline{\phi_R}\colon A_n(\bar{p}) \to A_n(\bar{q})$ in a similar fashion to ϕ_L and ϕ_R. Define $\overline{\phi_L} = \overline{L}^i(w)$, where i is the least non-negative integer such that $\overline{L}^i(w)$ contains no $\overline{p}$'s. Define $\overline{\phi_R} = \overline{R}^j(w)$, where j is the least non-negative integer such that $\overline{R}^j(w)$ contains no $\overline{q}$'s.

Lemma 2.5. *Let $w \in A_n(p)$ and $v \in A_n(q)$. We have*

$$\phi_R(v) = \overline{\overline{\phi_L}(\overline{v})} \tag{2}$$

$$\phi_L(w) = \overline{\overline{\phi_R}(\overline{w})}. \tag{3}$$

Intuitively, Equation (3) says the functions ϕ_L and $\overline{\phi_R}$ are conjugate under word reversal.

Example. Let $p = 011$ and $q = 001$, and let $w = 0001001$. We will show Equation (3) holds. An example above shows the computation of $\phi_L(w) = 0111011$. Next we evaluate $\overline{\overline{\phi_R}(\overline{w})}$. Firstly, we have $\overline{w} = 1001000$. Secondly, we evaluate $\overline{\phi_R}(\overline{w})$. Recall that $\overline{\phi_R}$ will scan right to left replacing $\overline{q}$'s with $\overline{p}$'s. The iterative replacement gives

$$\begin{aligned}
1001000 &\overset{\overline{R}}{\mapsto} 1001100 \\
&= 1001100 \overset{\overline{R}}{\mapsto} 1001110 \\
&= 1001110 \overset{\overline{R}}{\mapsto} 1101110 \\
&= 1101110.
\end{aligned}$$

This shows $\overline{\phi_R}(\overline{w}) = 1101110$. Since $\overline{1101110} = 0111011$, we have that $\overline{\overline{\phi_R}(\overline{w})} = 0111011$ as expected.

Proof of Lemma 2.5. We prove Equation (2) by induction on the number of replacement steps, denoted k. Then Equation (3) will follow by symmetry.

Let j be the number of steps required by ϕ_R applied to v. We set out to show

$$R^k(v) = \overline{\left(\overline{L}^k(\overline{v})\right)}, \tag{4}$$

for $0 \le k \le j$. It helps to first establish that, for any v that still has some p to replace, we have

$$R(v) = \overline{\left(\overline{L}(\overline{v})\right)}. \tag{5}$$

To see why this is true, observe that replacing the rightmost p is equivalent to

- reversing the word,
- replacing the leftmost $\bar{p}$, and then
- reversing again.

For the base case, the left-hand side of Equation (4) equals v because, when $k = 0$, there are no p's to replace in v. Similarly, $\overline{\big(\overline{L}^k(\overline{v})\big)} = \overline{(\overline{v})} = v$, because there are no $\bar{p}$'s to replace in $\overline{v}$.

Inductively, assume Equation (4) holds for some value of k where $0 \leq k < j$. We have

$$\begin{aligned}
\overline{\big(\overline{L}^{k+1}(\overline{v})\big)} &= \overline{\overline{L}\big(\overline{L}^{k}(\overline{v})\big)} \\
&= \overline{\overline{L}\big(\overline{R^{k}(v)}\big)} && \text{by the inductive hypothesis} \\
&= R\big(R^{k}(v)\big) && \text{using Equation (5)} \\
&= R^{k+1}(v).
\end{aligned}$$

This establishes Equation (4), which gives us Equation (2). □

3. The main theorem

With all this background, we are ready for the main result of the paper.

Theorem 3.1. *Let Σ be a finite alphabet, and let p and q be distinct, equal-length words on Σ. If the set of proper borders of p is equal to the set of proper borders of q, then $\phi_L \colon A_n(p) \to A_n(q)$ forms a bijection for all $n \in \mathbb{N}$.*

For example, the set of proper borders for each of the words 0100 and 0110 is $\{0\}$. On the other hand, 0110 and 1011 do not have the same set of proper borders, despite $b(0110) = \{1, 4\} = b(1011)$.

Remark. Let $w \in A_n(p)$. Observe that if w also avoids q then ϕ_L acts as the identity map on w. Therefore, Theorem 3.1 implies that words that avoid p and contain q are in bijection with words that avoid q and contain p.

A natural question is whether the number of q's in w is equal to the number of p's in $\phi_L(w)$. While this is usually the case, there do exist counterexamples. For example, let $p = 001$, $q = 110$, and $w = 1101110$. After 3 replacements, we see that $\phi_L(w) = 0000101$.

By Lemma 2.3, we have that ϕ_L is a map from $A_n(p)$ to $A_n(q)$. To prove Theorem 3.1, it suffices to show that ϕ_L is a bijection. To do this, we will show that ϕ_R is its inverse function, namely that $\phi_R\big(\phi_L(w)\big) = w$ for $w \in A_n(p)$ and also that $\phi_L\big(\phi_R(w)\big) = w$ for $w \in A_n(q)$. More specifically, we show that each one-step replacement L that takes place in $\phi_L(w)$ is undone by a one-step replacement R.

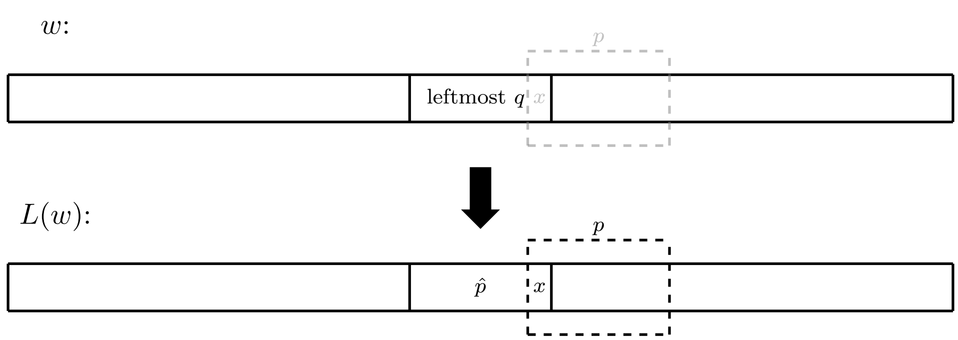
Proof of Theorem 3.1. Let $n \in \mathbb{N}$ and $w \in A_n(p)$. Let i be the number of steps required by ϕ_L applied to w. We will show by induction that $L^{k-1}(w) = R\big(L^k(w)\big)$ for all k satisfying $1 \le k \le i$, so that R is the left inverse of L. It will then follow that $\phi_R\big(\phi_L(w)\big) = w$. Let $v \in A_n(q)$; then

$$\begin{aligned}\phi_L\big(\phi_R(v)\big) &= \phi_L\big(\overline{\phi_L(\overline{v})}\big) \quad \text{by Equation (2)}\\ &= \overline{\phi_R\big(\phi_L(\overline{v})\big)} \quad \text{by Equation (3), letting } w = \big(\overline{\phi_L(\overline{v})}\big)\\ &= \overline{(\overline{v})} \quad \text{because } \overline{\phi_R} \text{ is the left inverse of } \overline{\phi_L}\\ &= v.\end{aligned}$$

Thus, we will have also shown that $\phi_L\big(\phi_R(v)\big) = v$, so that ϕ_R is both the left inverse and right inverse of ϕ_L. It will follow that $\phi_L\colon A_n(p) \to A_n(q)$ is a bijection.

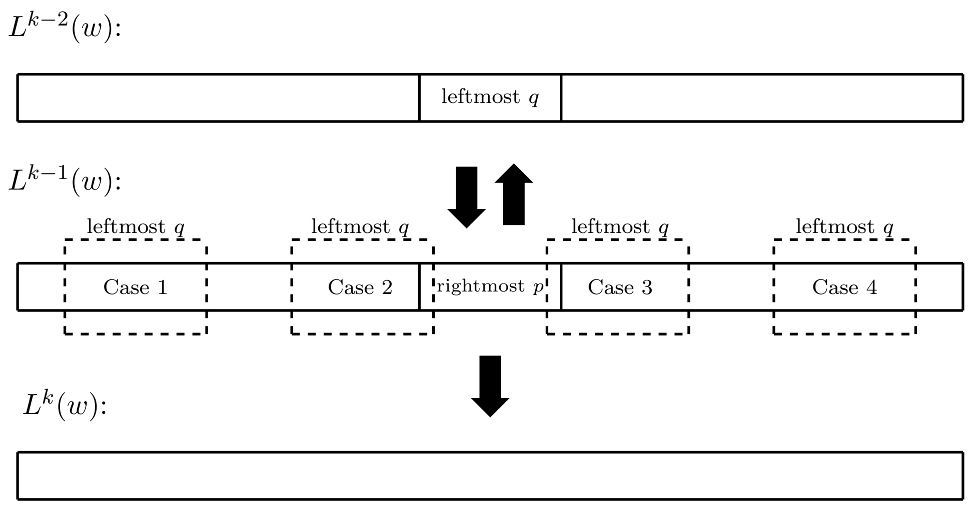
It remains to prove that $L^{k-1}(w) = R\big(L^k(w)\big)$. For the base case $k = 1$, the left-hand side of $L^{k-1}(w) = R\big(L^k(w)\big)$ is equivalent to applying zero L operations on w, so it trivially equals w. The right-hand side of this equation is $R\big(L(w)\big)$. We denote the new p inserted by L as $\hat{p}$. We claim that $\hat{p}$ is the rightmost p in $L(w)$; then the R step function will find it first and will replace $\hat{p}$ back with a q.

To prove the claim, assume that $\hat{p}$ is not rightmost in $L(w)$. Then there is a p to the right of $\hat{p}$. If this p does not overlap $\hat{p}$, then it would have also been present in w. But w is p-avoiding; therefore p must overlap $\hat{p}$. We denote the overlap in $L(w)$ as x:



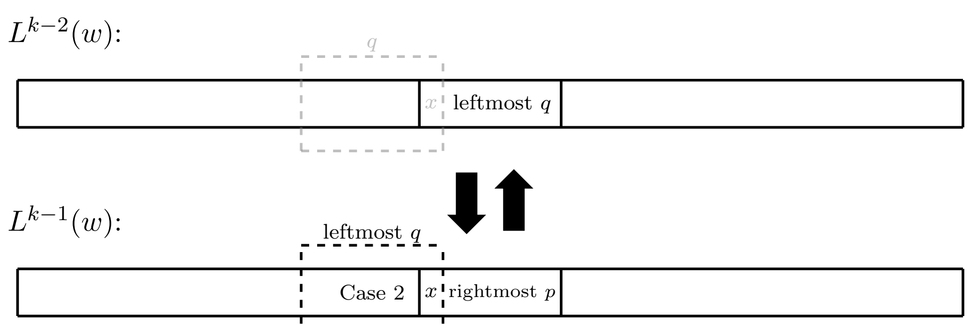
The overlap x is a border of p. Observe that since p and q have the same borders, the border segment x is also in w as a suffix of q. This means that x wasn't altered when we swapped in $\hat{p}$. This implies that the overlapping p is also in w. This contradicts our assumption that w is p-avoiding. Therefore, $\hat{p}$ is the rightmost p in $L(w)$, implying that $L^{k-1}(w) = R\big(L^k(w)\big)$ holds for $k = 1$.

Inductively, assume that $L^{k-2}(w) = R\big(L^{k-1}(w)\big)$ for some k between 1 and the number of steps required by ϕ_L. This assumption means that once we replace the leftmost q in $L^{k-2}(w)$ with p, this new p must be the rightmost p in $L^{k-1}(w)$ because we assumed that the R function maps $L^{k-1}(w)$ back to $L^{k-2}(w)$ (the R function scans from right to left); this is indicated by an arrow in each direction in the diagram below. To show the inductive hypothesis holds for $k+1$, we need to show this same relationship holds between words $L^{k-1}(w)$ and $L^k(w)$. Thus, we wish to show that, once we replace the leftmost q in $L^{k-1}(w)$ with p, this new p in $L^k(w)$ is the rightmost. The proof is split into four cases based on the possible positions of the leftmost q in the word $L^{k-1}(w)$.

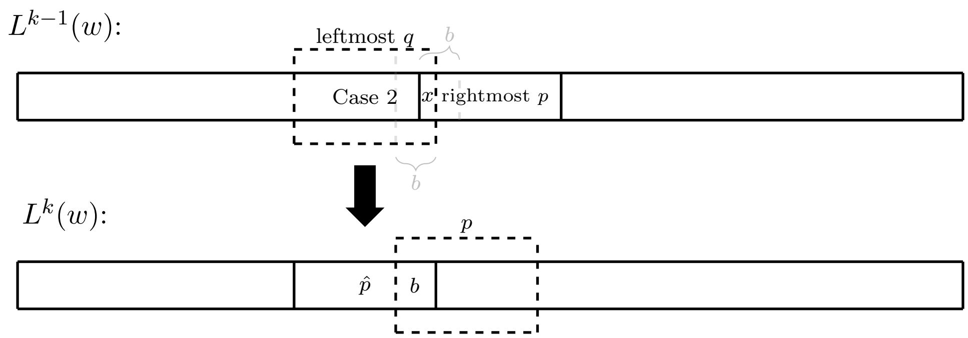


Case 1. This position of q in $L^{k-1}(w)$ implies that there is a q in the same position in $L^{k-2}(w)$. But then we have a q to the left of the leftmost q in $L^{k-2}(w)$, a contradiction.

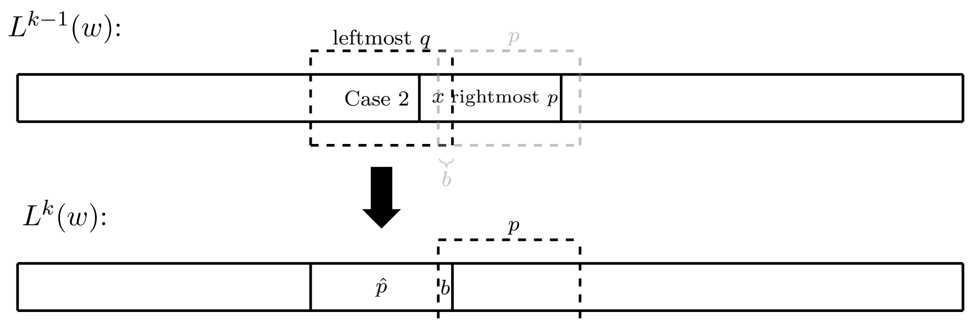
Case 2. Let x be the overlap of the leftmost q and the rightmost p in $L^{k-1}(w)$. Suppose first that x is a border of q. We use a similar argument as in the base case. Since x did not change from $L^{k-2}(w)$ to $L^{k-1}(w)$, there must exist a q in the same spot in $L^{k-2}(w)$. This q is left of the leftmost q in $L^{k-2}(w)$, so we have a contradiction.



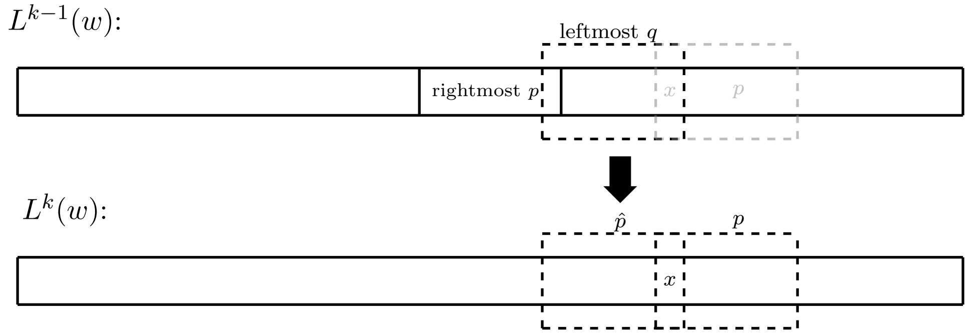
Suppose instead that x is not a border of q and therefore not a border of p. Toward a contradiction, assume that $\hat{p}$ is not the rightmost p in $L^k(w)$. For this to occur, $\hat{p}$ must be overlapping with another p on its right. We label this overlap segment b. Observe that b is a border of p and q. In particular, b is a suffix of the leftmost q and a prefix of the rightmost p in $L^{k-1}(w)$. If $|b| > |x|$, it would follow that x is both a prefix and suffix of b. This would imply x is a border of p, a contradiction.



So it must be that $|b| < |x|$. Notice that the overlapping p in $L^k(w)$ must have existed in the same position in $L^{k-1}(w)$ since b was left unchanged when $\hat{p}$ was swapped in. This puts a p to the right of the rightmost p in $L^{k-1}(w)$, another contradiction.



Case 3. Suppose for contradiction that $\hat{p}$ is not the rightmost p in $L^k(w)$. Then $\hat{p}$ must overlap with another p to its right. We label the overlap x in the diagram below. Since x is a border of p, it is a border of q. Hence x was left unchanged when $\hat{p}$ was substituted in. This implies that the p to the right of $\hat{p}$ must have existed in the previous word $L^{k-1}(w)$. But this puts a p to the right of the rightmost p in $L^{k-1}(w)$, a contradiction.



Case 4. Case 4 follows using the same argument as in Case 3. □

We contextualize the proof with an example and a counterexample.

Example. Let $p = 0110$ and $q = 0010$. Note that the set of proper borders for both p and q is $\{0\}$, so ϕ_L is a bijection from $A_n(p)$ to $A_n(q)$. Let $w = 1001001011 \in A_{10}(p)$. This example demonstrates how each single scan function L is undone by the function R. Observe that the first replacement aligns with the base case of the proof for Theorem 3.1, while the second replacement aligns with Case 3B. Running ϕ_L on w gives

$$\begin{aligned} 1001001011 &\overset{L}{\mapsto} 1011001011 \\ &= 1011001011 \overset{L}{\mapsto} 1011011011 \\ &= 1011011011. \end{aligned}$$

Now we will run $\phi_L(w) = 1011011011$ through ϕ_R to see that we get w back. We also see that single scan R successfully undoes every replacement made by an L. This gives us

$$\begin{aligned} 1011011011 &\overset{R}{\mapsto} 1001011011 \\ &= 1001011011 \overset{R}{\mapsto} 1001001011 \\ &= 1001001011 = w. \end{aligned}$$

Example. We now present a short counterexample. Let $p = 1011$ and $q = 0100$. Note that $b(p) = \{1, 4\} = b(q)$, but 1 is a proper border of p and not a proper border of q. So, Theorem 3.1 does not guarantee ϕ_L will form a bijection from $A_n(p)$ to $A_n(q)$. For the word $w_1 = 0101011 \in A_7(p)$, we have

$$0101011 \overset{L}{\mapsto} 0100100 = 0100100.$$

For another word $w_2 = 1011100 \in A_7(p)$, we have

$$1011100 \overset{L}{\mapsto} 0100100 = 0100100.$$

Observe that $\phi_L(w_1) = 0100100 = \phi_L(w_2)$ so that ϕ_L does not provide a bijection.

3.1. **How many bijections do we obtain?** One might wonder how many of the possible bijections ϕ_L provides. We know that ϕ_L forms a bijection from $A_n(p)$ to $A_n(q)$ if $b(p) = \{|p|\} = b(q)$. Words with no proper borders, such as these, are known as *borderless* words. The density of borderless words on a finite alphabet has been analyzed in detail. Silberger [7] first discovered a recursive formula to count borderless words, and Holub & Shallit [4] investigated the probability that a random word is borderless. Notably, a long binary word p chosen randomly has $\approx 27\%$ chance of being borderless and $\approx 30\%$ chance of having the border length set $\{1, |p|\}$. The function ϕ_L provides a bijection for all borderless pairs and almost half of the pairs whose border length set is $\{1, |p|\}$. These cases alone account for a sizable chunk of possible avoidant-equivalent word pairs, which is why the percentage of pairs for which we have natural bijections is so high.

Pattern length	ϕ_L bijection pairs	Composition bijection pairs	Equivalent pairs
1	1	1	1
2	1	2	2
3	6	8	8
4	21	32	32
5	88	120	120
6	312	460	460
7	1212	1708	1716
8	4649	6764	6780
9	18264	26072	26168
10	71058	103460	103764
11	279946	403836	405404
12	1107836	1613132	1618556

TABLE 1. Summary of bijections between patterns on $\{0, 1\}$.

Table 1 contains data on the number of pairs of patterns on $\Sigma = \{0, 1\}$ for which we have a natural bijection. The "Equivalent pairs" column gives the total number of unordered pairs of patterns p and q for which $b(p) = b(q)$. The second column counts pairs for which ϕ_L establishes a bijection. Additionally, if we allow compositions with the reversal function and letter permutation function, we are able to obtain even more bijections; these pairs are counted in the third column.

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